

THE NICHOLS ALGEBRA OF SCREENINGS

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ABSTRACT. Two related constructions are associated with screening operators in models of two-dimensional conformal field theory. One is a local system constructed in terms of the braided vector space X spanned by the screening species in a given CFT model and the space of vertex operators Y and the other is the Nichols algebra $\mathfrak{B}(X)$ and the category of its Yetter–Drinfeld modules, which we propose as an algebraic counterpart, in a “braided” version of the Kazhdan–Lusztig duality, of the representation category of vertex-operator algebras realized in logarithmic CFT models.

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1. INTRODUCTION

We discuss how braided categories — more specifically, representation categories of braided Hopf algebras — can be associated with nonsemisimple (logarithmic) models of conformal field theory (CFT). The subject of braided categories related to CFT has been profoundly investigated in [1, 2, 3], but considering the problem at a somewhat different angle may not be altogether unworthy. The proposal in this paper is to take the algebra of screening operators, regard it as a braided Hopf algebra, and then construct a category of its Yetter–Drinfeld modules from other CFT data, the vertex operators in a given model. The algebras generated by screening operators are in fact Nichols algebras [4, 5, 6, 7, 8, 9, 10, 11, 12]—universal braided Hopf algebra quotients of tensor algebras that have received considerable attention recently, originally motivated by Andruskiewitsch and Schneider’s classification program (see [13, 14, 15] and the references therein). We here extend their use to a bridge between CFT and braided categories.

Nichols algebras. Under the name of “bialgebras of type one,” Nichols algebras (more precisely, their bosonizations) originally appeared in [4]. They have several definitions, whose equivalence is due to [16] and [5] (where they still feature under a more indigenous name). In addition to the papers cited above, they also appeared in [17, 18, 17, 19, 20, 16, 21, 22] (see [23] and the references therein for recent progress).¹

The Nichols algebra $\mathfrak{B}(X)$ of a braided linear space X is a graded braided Hopf algebra, $\mathfrak{B}(X) = \bigoplus_{n \geq 0} \mathfrak{B}(X)^{(n)}$ (a vector-space direct sum), such that $\mathfrak{B}(X)^{(0)}$ is merely the ground field and $\mathfrak{B}(X)^{(1)} = X$, and this last space has two properties: it coincides with the

¹An important technicality, which we nevertheless tend to ignore, is that there is a distinction between quantum symmetric algebras [21] and Nichols algebras proper; the latter are selected by the condition that the braiding is *rigid* (which in particular guarantees that the duals X^* are objects in the same braided category with the X); see [24, 25].

space of *all primitive elements* $P(X) = \{x \in \mathfrak{B}(X) \mid \Delta x = x \otimes 1 + 1 \otimes x\}$ and it *generates all of* $\mathfrak{B}(X)$ as an algebra.

The Nichols algebras occurred independently in [26], for the purposes of constructing a quantum differential calculus and in a form suggestive of their role of “fully braided generalizations” of symmetric algebras, viz.

$$\mathfrak{B}(X) = k \oplus X \oplus \bigoplus_{n \geq 2} X^{\otimes n} / \ker \mathfrak{S}_n,$$

where \mathfrak{S}_n is the total braided symmetrizer. This gives another characterization of Nichols algebras.

Finally, the Nichols algebra of a linear braided space X can be defined in terms of a duality pairing $\langle \cdot, \cdot \rangle : T(X^*) \otimes T(X) \rightarrow k$. It follows that $\mathfrak{B}(X) = T(X)/I(X)$, where $I(X)$ is the kernel of the pairing [5].

Motivation. Nichols algebras are a core element in the construction of the nilpotent part of deformed enveloping algebras $\mathcal{U}_q^+(\mathfrak{g})$ [17] (also see [27, 21]). This last development has a well-known “physical” reformulation, stating that quantum Serre relations are satisfied by screening operators in CFT [28] (also see the references therein, [29] in particular). In a conformal model, the action of screenings on vertex operators does not yield elements of the vertex-operator algebra of that model in general, but generates a larger structure, whose properties have not been studied much just because it is not a vertex-operator algebra and, in addition, carries a subtle dependence on the choice of contours involved in the definition of screenings. Objects of this sort, sometimes called screened (or dressed) vertex operators, are indispensable, however, in constructing correlation functions in CFT [30]. We use them in what follows to construct the categories of Hopf bimodules and Yetter–Drinfeld modules over Nichols algebras of screenings.

This construction of (co)modules over an algebra of screenings is closely related to a more geometric part of this paper—a local system (or, a flat bundle) on the configuration space $\mathbb{X}_{m,n}$ of n points on an m -punctured complex plane. The points are to be thought of as positions of screenings, and the punctures as the positions of vertex operators. The latter are fixed, and the former, in physical terms, are “integrated over” along some contours. The idea of screening operators being represented by iterated contour is refined by a “space–time” decomposition of the complex plane, with the integrations running in the “space” direction, with which we associate a stratification of the $\mathbb{X}_{m,n}$ configuration space of screenings. The stratification readily leads to the local system mentioned above, and the homology complex with coefficients in this local system gives rise to a shuffle algebra, in which the Nichols algebra $\mathfrak{B}(X)$ is a subalgebra. The local system becomes an abundant source of Yetter–Drinfeld $\mathfrak{B}(X)$ -modules.

Outline of the construction. In most of the paper, we work with much more general braided spaces and categories than those observed in CFT models, and only specialize to

diagonal braiding (and then, actually, to a very particular case) in the Sec. 5.

Screening operators, in the way they act on vertex operators, are given by contour integrals in one form or another, and are in this sense “nonlocal.” We abstract the contour integral representation to a family parallel lines—“spatial directions on the complex plane”—drawn through each puncture (solid lines in Fig. 1.1). We also draw arbitrarily

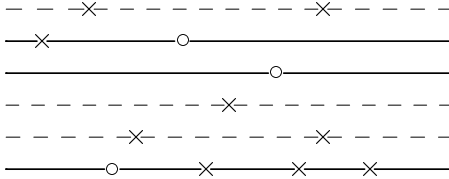


FIGURE 1.1. “Fixed” parallel lines are drawn through the punctures (\circ). Additional lines can move in between so as to remain parallel and not collide with or pass through one another or the fixed lines. All lines are populated with screenings (\times).

many additional parallel lines, not running through the punctures, and populate them with screenings. Once two screenings (shown by crosses in Fig. 1.1) are on the same line, the notion of precedence is clearly defined, and hence so is the notion of the *transposition* of any two screenings via braiding. The braiding is inherited from a given CFT model as a map

$$\Psi : X \otimes X \rightarrow X \otimes X$$

that satisfies the braiding equation, where X is the linear span of the different screening species. Somewhat more abstractly—without direct reference to CFT—we just assume that a braided linear space X is associated with every cross in a picture such as in Fig. 1.1.

Placing n crosses on the system of lines on an m -punctured complex plane, as in Fig. 1.1, defines a stratification of the configuration space of crosses/screenings $\mathbb{X}_{m,n}$. A stratum—“cell”—is defined by any particular distribution of n crosses over such a system of lines, with the movable lines (from 0 to n in number) allowed to move so as to remain parallel and not collide with or pass through one another or the fixed lines, and the crosses allowed to slide along the lines so as to not collide with one another or with the punctures and not pass through the punctures.

With each puncture, we associate another braided space Y (in CFT, a linear span of vertex operators), and let Ψ denote the braiding in all cases $X \otimes Y \rightarrow Y \otimes X$, etc. Notably, an essential part of our construction does *not* require the braiding $Y \otimes Y \rightarrow Y \otimes Y$.

Given X and Y , we define a *local system* (the space of sections of a flat bundle) over the configuration space \mathbb{X}_m of (noncoincident and indistinguishable) points on the complex plane with m punctures. For a fixed number n of points, we take all cells, i.e., all possibilities to distribute n crosses over $m + \ell$ lines, $0 \leq \ell \leq n$. A braided linear space linearly

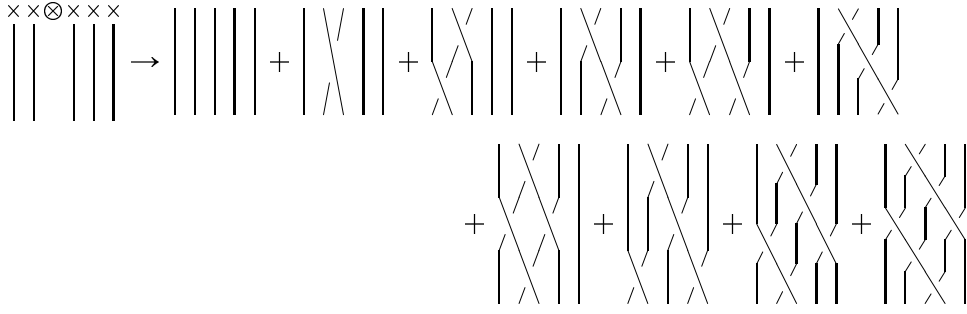


FIGURE 1.2. The $X^{\otimes r} \otimes X^{\otimes s} \rightarrow X^{\otimes(r+s)}$ shuffle product (with $r = 2$ and $s = 3$ in the picture) involves $\binom{r+s}{r}$ terms, each of which represents a braid group element whose action arranges the crosses “fed in” to the top of each strand (this and other braid diagrams are to be read from top down). Each term shows precisely how the $2 + 3$ crosses are sent into their “target” positions.

isomorphic to $X^{\otimes n} \otimes Y^{\otimes m}$ is then associated with each cell. The local system is defined by specifying how these spaces are identified under the “restriction map” associated with taking the boundary of each cell.

Taking the boundary means that a movable line in Fig. 1.1 merges with another (movable or fixed) line, and the crosses carried by the two lines are “collectivized.” For example, two movable lines

$$\begin{array}{cccccc} - & - & \times & - & \times & - & - \\ - & \times & - & \times & - & \times & - & - \end{array} \quad \begin{array}{l} (r \text{ crosses}) \\ (s \text{ crosses}) \end{array}$$

merge into

$$- \times - \times - \times - \times - \times - \quad (r + s \text{ crosses})$$

The restriction maps are naturally constructed in terms of the braiding, and, expectedly, are given by “quantum” shuffles [21] (see Appendix A). In a merger of two movable lines, for example, the crosses are collectivized by (r, s) -shuffle permutations lifted to the braid group algebra. This is illustrated in Fig. 1.2.

This “merger” operation is associative and is part of a braided bialgebra structure [21]. The coproduct in this “braided Hopf algebra of crosses” is given by splitting each line into two and, for a line with j crosses, taking the sum over all the $j + 1$ possibilities of distributing the j indistinguishable crosses between two lines,

$$(1.1) \quad \begin{array}{ccccccc} - & \times & - & \times & - & \times & - \\ - & \times & - & \times & - & \times & - \end{array} \longrightarrow \begin{array}{ccccccc} \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times \end{array} \longrightarrow$$

$$\begin{array}{ccccccc} - & \times & - & \times & - & \times & - \\ - & \times & - & \times & - & \times & - \end{array} + \begin{array}{ccccccc} - & \times & - & \times & - & \times & - \\ - & \times & - & \times & - & \times & - \end{array} + \begin{array}{ccccccc} - & \times & - & \times & - & \times & - \\ - & \times & - & \times & - & \times & - \end{array} + \begin{array}{ccccccc} - & \times & - & \times & - & \times & - \\ - & \times & - & \times & - & \times & - \end{array}$$

(the “deconcatenation” coproduct). The product and coproduct define a braided bialgebra [21]. *The antipode given by orientation reversal* makes it into a braided Hopf algebra, $\mathcal{H}(X)$. Those r -cross elements

$$(1.2) \quad - \times - \times - \times - \times -$$

that can be obtained from a single-cross one by iterated multiplication span the Nichols algebra $\mathfrak{B}(X) \subset \mathcal{H}(X)$.

The fixed lines (those with a puncture each), furthermore, become elements of $\mathcal{H}(X)$ (and $\mathfrak{B}(X)$) bi(co)modules. The left and right *actions* of elements of $\mathcal{H}(X)$ (movable lines, such as (1.2)) on fixed lines (such as $\times - \times - \circ - \times$) amount to distributing the “new” crosses over the entire fixed line, again by quantum shuffles, as is illustrated in Fig. 1.3 (where the visualization conventions are somewhat different from those in Fig. 1.2). The left and right *coactions* are by deconcatenation of the crosses from the half-line on the respective side of the puncture. The resulting bimodule bicomodule actually turns out to be a Hopf bimodule (“bi-Hopf module,” “bicovariant bimodule”) over $\mathcal{H}(X)$.

FIGURE 1.3. The left action \cdot of “the Hopf algebra of crosses” on its module. The arrows indicate the braiding used to put the “new” cross into the different positions. (Alternatively, the braiding can be represented in terms of diagrams of the type of those in Fig. 1.2.) Under the right action, the “new” crosses populate the fixed line starting from the right.

From Hopf bimodules, we pass to the equivalent category [31] of Yetter–Drinfeld modules. The (left–left) Yetter–Drinfeld modules are given by right coinvariants in Hopf bimodules, which in our case means taking fixed lines with no crosses to the right of the puncture,

$$(1.3) \quad - \times - \times - \times - \circ -$$

They carry the left “adjoint” action of the braided Hopf algebra, as is illustrated in Fig. 1.4.

We next allow certain degenerations by considering $\mathfrak{B}(X)$ -(co)modules that are (subspaces in) $\bigoplus_{i,j \geq 0} X^{\otimes i} \otimes Y \otimes X^{\otimes j} \otimes Y$, with two (or more) punctures on the same line:

$$- \times - \circ - \times - \times - \circ -$$

Because the punctures are associated with positions of vertex operators in CFT, such modules are referred to as two-vertex (or multivertex) modules.

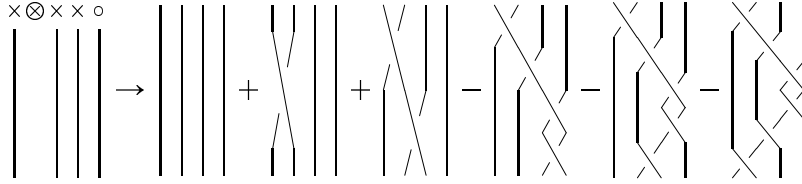


FIGURE 1.4. An example of the left adjoint action. A single “new” cross populates a fixed line with two crosses as is described by the braid diagrams. The cross arrives to each of the three possible positions in two ways (one with the plus and the other with the minus sign in front). That the cross never stays to the right of the puncture is a manifestation of the fact that the space of right coinvariants in a Hopf bimodule is invariant under the left adjoint action.

The multivertex modules carry the following $\mathcal{H}(X)$ (and $\mathcal{B}(X)$) coaction and action: the deconcatenation coaction *up to the first puncture* and the “cumulative” *left adjoint action*, meaning that all punctures except the rightmost one are viewed on equal footing with the crosses (an exact definition is given in 4.2.3). Remarkably, the multivertex modules are Yetter–Drinfeld modules. We also evaluate the effect of exchanging the two punctures in the two-vertex modules, i.e., the “fusion-product braiding.” This operation is important, in particular, because its square commutes with “everything” and is therefore related to the center of the category.²

When it comes to CFT-related applications, the braiding Ψ reduces to a diagonal one.³ It is convenient to first specialize to the case where Ψ is the braiding in some ${}^H_H\mathcal{YD}$, the category of Yetter–Drinfeld modules over an ordinary Hopf algebra H . A braided Hopf algebra whose braiding is rigid can be realized as a Hopf algebra in the category of Yetter–Drinfeld modules over an ordinary Hopf algebra H [39], although H is by far not unique (but the Nichols algebra depends only on the braiding, not on the Yetter–Drinfeld structure; Nichols algebras have indeed been extensively studied in the Yetter–Drinfeld setting).

In actual CFT applications in this paper, we restrict ourself to the simplest case of a one-dimensional space X , with the braiding whose associated total symmetrizer vanishes on p -fold tensor products. This is the setting of the celebrated $(p, 1)$ logarithmic CFT models [40, 41, 42, 32, 43, 44, 45, 46], which, their already long history notwithstanding, are currently being studied from various standpoints. (In addition to the works just cited, we also refer the reader to [47, 48, 49, 50, 51, 52, 53, 54, 55, 56] and the references therein

²In CFT applications, the center is generally expected to be isomorphic to the space of torus amplitudes in the corresponding logarithmic CFT (isomorphic as both commutative associative algebras and $SL(2, \mathbb{Z})$ representations; cf. [32] and [33, 34]).

³Diagonal braiding has been the subject of considerable activity in the study of Nichols algebras [35, 36, 8, 12, 37, 9, 38, 23].

for the various aspects of logarithmic conformal field theories.) We show how the general constructions defined in terms of “quantum shuffles” specialize to the particular case of diagonal braiding associated with $(p, 1)$ logarithmic CFT models. The general construction of Yetter–Drinfeld $\mathfrak{B}(X)$ -modules specializes to the corresponding Nichols algebra. For the purposes of comparison with logarithmic CFT, the modules thus constructed are then to be viewed *not* as objects in the braided category of Yetter–Drinfeld $\mathfrak{B}(X)$ -modules but as “just” $\mathfrak{B}(X)$ module comodules either without the braiding at all or with the monodromy (“squared braiding”) structure (the morphisms are then respectively those of the module comodule structure or of the module comodule plus monodromy structure). Thus modified, the Yetter–Drinfeld $\mathfrak{B}(X)$ modules then conjecturally provide an equivalence with representations of the triplet W algebra in the $(p, 1)$ model (also viewed as either an Abelian category or a category with a monodromy structure; we note that monodromy is intrinsically defined in CFT).

Remarks on the known logarithmic Kazhdan–Lusztig duality. Ordinary (nonbraided) Hopf algebras (“factorizable ribbon, although not quasitriangular, quantum groups”) have previously been proposed to capture (to different degree) the important features of representation categories of vertex-operator algebras in logarithmic CFT models [32, 43, 33, 34, 44, 46, 57, 58]. We expect the contact with this earlier development to be achieved in terms of *bosonization* [59], or in fact double bosonization [60, 61], of braided Hopf algebras. Bosonization, which turns “braided statistics” into “bosonic statistics” where objects are transposed at no expense, has been known under the name of Radford’s biproduct since [62], before the actual invention of braided Hopf algebras [63]. From a Hopf algebra $\mathcal{H} \in {}^H_H\mathcal{YD}$, the Radford biproduct produces an ordinary Hopf algebra based on the smash product $\mathcal{H} \# H$.

The quantum groups featuring in the known instances of logarithmic Kazhdan–Lusztig duality have been arrived at using a Drinfeld double of an ordinary Hopf algebra generated by screening(s) and a grading operator. The procedure, despite its reasonable success in capturing the properties of relevant CFT models, was nevertheless somewhat ad hoc. It would be quite interesting to see how the previous proposal can be derived from our braided setting in this paper by a version of double bosonization for the particular, rather special, quantum groups. It is worth noting here that for an ordinary Hopf algebra H , the Drinfeld double $\mathcal{D}(H)$ answers the question of describing H -module comodules that satisfy the Yetter–Drinfeld axiom as objects in a monoidal category of *modules* over *something*. The answer is that *something* = $\mathcal{D}(H)$ (the category of $\mathcal{D}(H)$ representation is in fact equivalent to the category of Yetter–Drinfeld H -modules [26, 64]). In the braided case, an attempted Drinfeld double construction “tangles up” and does not work unless major simplifying assumptions are made.⁴ In the general case, one is left with

⁴A “nonobvious” construction of a “braided Drinfeld double” was given in [65].

just Yetter–Drinfeld modules, which take over the role of representations of the quantum groups occurring in the “nonbraided” logarithmic Kazhdan–Lusztig duality.

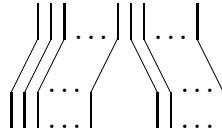
Remark on diagrams. There are *two types of diagrams* in this paper: (i) the standard diagrams used in the theory of braided (Hopf $|\mathbf{bi}| \dots$) algebras (see, e.g., [66]), and (ii) braid diagrams (i.e., those *representing elements of the braid group algebra*). These are similar, but not identical in meaning. The similarity is that



represents an instance of braiding in either case. In the braided-Hopf-algebra language, the lines can represent two braided linear spaces, whereas in terms of the braid group algebra, these are just the two strands of the simplest nontrivial braid. The difference is that, for example, the coproduct on a braided Hopf algebra \mathcal{H} is represented in braided-algebra language as



In what follows, we deal with a *specific* \mathcal{H} , the one constructed on the tensor algebra of some braided linear space X . The braid diagrams are then those where each strand corresponds a copy of this X . In terms of braid-group diagrams, the above coaction diagram translates into a *sum* of diagrams of the form



(that the two bunches are assigned to two different copies of \mathcal{H} is, strictly speaking, a “decoration” of this trivial braid diagram).

A characteristic example of braided-algebra diagrams is provided by (B.7) and (B.6) (page 53), and that of braid diagrams, by Fig. 1.4 (page 7). That figure in fact shows how (B.6) specializes to the braid group algebra.

2. SCREENING CONFIGURATIONS AS A STRATIFICATION

Let $\mathbb{C}_{w_1, \dots, w_m}$ be the complex plane punctured at m points w_1, \dots, w_m . We consider the configuration space of (an arbitrary number of) indistinguishable, pairwise noncoincident points (“screenings”) on $\mathbb{C}_{w_1, \dots, w_m}$. The space of such configurations is

$$\mathbb{X}(w_1, \dots, w_m) = \bigsqcup_{n>0} \mathbb{X}_n(w_1, \dots, w_m),$$

$$\mathbb{X}_n(w_1, \dots, w_m) = \left(\underbrace{\mathbb{C}_{w_1, \dots, w_m} \times \mathbb{C}_{w_1, \dots, w_m} \times \dots \times \mathbb{C}_{w_1, \dots, w_m}}_n \setminus \Delta \right) / \mathbb{S}_n,$$

where $\setminus \Delta$ means the removal of all subdiagonals (the points are noncoincident), and $/\mathbb{S}_n$ is the quotient by the action of the symmetric group on n elements (the points are indistinguishable). We call the points “crosses” in accordance with the chosen way of their pictorial representation. In contrast to the punctures, which are fixed, the crosses are *movable* because neither their positions on any line, nor the positions of the movable lines are fixed.

We fix two braided linear spaces X and Y and associate a copy of Y with each puncture and a copy of X with each cross.

The aim of this section is to construct a local system $\mathcal{L}_m = \mathcal{L}(w_1, \dots, w_m)$ whose associated flat bundle, with the fiber linearly isomorphic to $X^{\otimes n} \otimes Y^{\otimes m}$, generalizes the bundle of conformal blocks, with a Knizhnik–Zamolodchikov-like flat connection. We begin with introducing a stratification of $\mathbb{X}_n(w_1, \dots, w_m)$ based on a “space–time” decomposition of the complex plane. We write $\mathbb{X}_{m,n} = \mathbb{X}_n(w_1, \dots, w_m)$ for brevity.

2.1. Local system. For a fixed m , we represent each $\mathbb{X}_{m,n}$ as

$$\mathbb{X}_{m,n} = \bar{F}^{2n} \supset \bar{F}^{2n-1} \supset \dots \supset \bar{F}^n,$$

where the \bar{F}^k are dimension- k strata, \bar{F}^k being closed in \bar{F}^{k+1} , each consisting of several connected components (also called strata):

$$\bar{F}^k = \bigcup_{j \in I_k} \bar{F}_j^k, \quad k = n, \dots, 2n,$$

where I_k are some finite sets (for example, $|I_{2n}| = \binom{n+m}{n}$).

We actually construct *open* contractible subspaces $F_{j_k}^k$ whose closure in F^{k+1} is the corresponding connected stratum, $\overline{F_j^k} = \bar{F}_j^k$, and the boundary of each F_j^k is contained in $\bigcup_i F_i^{k-1}$. Then

$$\mathbb{X}_{m,n} = \bigcup_{n \leq k \leq 2n} \bigcup_{j_k \in I_k} F_{j_k}^k,$$

We call the $F_{j_k}^k$ “cells.”

Construction of the F_j^k is described in **2.1.1**. For a given F_a^k , the F_b^{k-1} such that the codimension-1 part of ∂F_a^k is contained in $\bigcup_{b \in I_{k-1}} F_b^{k-1}$ are described in **2.1.2**.

2.1.1. Constructing the cells. All cells F_j^k are obtained by simply placing n crosses on a family of parallel lines (“spatial sections”) and then hierarchically restricting the possible configurations of the lines (see Fig. 1.1).

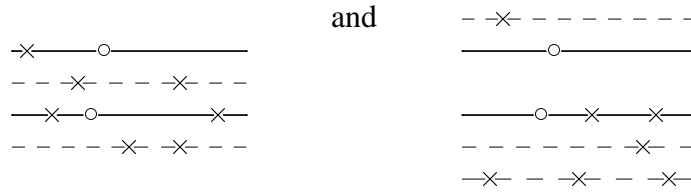
- (0) Select, once and for all, a straight line on $\mathbb{C}_{w_1, \dots, w_m}$ and consider the family of parallel lines, assuming that by the choice of the first line, none of the lines passes through more than one puncture. An orientation must also be selected, the same

on all lines. In addition, the lines must be ordered inside the family (“the global time”).⁵

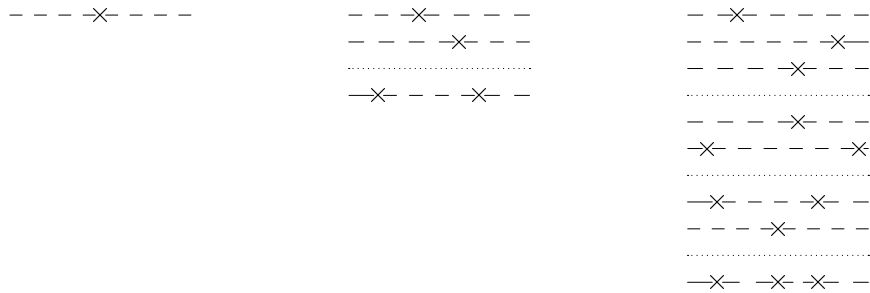
- (1) Draw a line from the family passing through each puncture and label these m lines “monotonically” in accordance with the order chosen in the family. These m lines are called *fixed lines* in what follows.
- (2) Draw r , $0 \leq r \leq n$, lines from the family not passing through any puncture; these lines are not fixed but can move so as to remain parallel and to not collide with or pass through one another or any of the fixed lines. Place at least one cross on each of these movable lines.

A cell is by definition a piece of $\mathbb{X}_{m,n}$ consisting of all configurations where the crosses are placed on fixed and movable lines and are allowed to slide along the lines so as to *not collide with one another and not pass through the punctures*, and the movable lines move as just described. The (real) dimension of a cell constructed this way is $k = n + r$.

2.1.1.1. Example. Two examples of cells in $\mathbb{X}_{2,7}$ are represented by the configurations



2.1.1.2. Example On a *nonpunctured* plane, all cells are enumerated following the simple pattern (with dotted lines separating different cells)



with the columns respectively corresponding to $\mathbb{X}_{0,1}$, $\mathbb{X}_{0,2}$, $\mathbb{X}_{0,3}$, and so on (2^{n-1} cells in the n th column).

2.1.1.3. Example There are $\binom{n+m}{n}$ cells of the maximum dimension $2n$. To be more specific, we can assume that the punctures w_1, \dots, w_m are chosen such that $a_1 \equiv \text{Im } w_1 <$

⁵A possible way to do this is to take the lines to be level lines of the function $f(z) = \alpha \text{Re}(z) + \beta \text{Im}(z)$ with real α and β ; some other choices of the function suggest interesting generalizations.

$a_2 \equiv \text{Im } w_2 < \cdots < a_m \equiv \text{Im } w_m$. In \mathbb{R}^{2n} with coordinates x_i, y_i , $i = 1, \dots, n$, the list of maximum-dimension cells is then given by

$$\begin{aligned}
& -\infty < y_1 < y_2 < \cdots < y_n < a_1 < \cdots < a_m < +\infty, \\
& -\infty < y_1 < y_2 < \cdots < y_{n-1} < a_1 < y_n < \cdots < a_m < +\infty, \\
& -\infty < y_1 < y_2 < \cdots < y_{n-1} < a_1 < a_2 < y_n < \cdots < a_m < +\infty, \\
& \vdots \\
& -\infty < y_1 < y_2 < \cdots < a_1 < y_{n-1} < a_2 < \cdots < a_{m-1} < y_n < a_m < +\infty, \\
& \vdots \\
& -\infty < a_1 < \cdots < a_m < y_1 < y_2 < \cdots < y_n < +\infty
\end{aligned}$$

with $-\infty < x_i < +\infty$, $1 \leq i \leq n$, in each row (each cell). This explicitly shows that the cells are open neighborhoods in \mathbb{R}^{2n} .

It follows that the dimension- k cells, $n \leq k \leq 2n$, are in bijective correspondence with the data

$$(2.1) \quad ([j_1], [j_2], \dots, [j_{k+m-n}]),$$

where

- (1) each $[j_a]$ is either an integer $j_a > 0$ or a pair of integers $(j'_a \geq 0, j''_a \geq 0)$,
- (2) there are exactly m pairs of integers among the $[j_a]$,
- (3) $\sum_{a=1}^{k+m-n} [j_a] = n$, with each summand understood as either j_a or $j'_a + j''_a$ depending on the type.

In (2.1), each $[j_a] = j_a$ represents a movable line carrying j_a crosses, and each $[j_a] = (j'_a, j''_a)$ represents a fixed line carrying j'_a crosses before and j''_a crosses after the puncture (relative to the chosen orientation). The *total* number of lines is $k + m - n$, of which $k - n$ are movable lines.

For fixed punctures, any cell represented as in (2.1) is uniquely assigned to one of the $\mathbb{X}_{m,n}$ spaces: the number of punctures m is just the number of pairs of integers among the $[j_a]$, and the number of crosses is $n = \sum_{a=1}^{k+m-n} [j_a]$, as noted above. This allows us to omit explicit indications of the $\mathbb{X}_{m,n}$ space to which a given cell belongs.

The two cells in **2.1.1.1** are thus written as $((1, 0), 2, (1, 1), 2)$ and $(1, (0, 0), (0, 2), 1, 3)$.

Oriented cells are $([j_1], [j_2], \dots, [j_s]; \pm 1)$.

2.1.2. The F_b^{k-1} in the closure of F_a^k : line merger. We now (indirectly) describe all pairs (F_a^k, F_b^{k-1}) of a dimension- k and a dimension- $(k-1)$ cell such that the codimension-1 part of ∂F_a^k contains F_b^{k-1} . This amounts to describing all cases of a merger of either two neighboring movable lines or a movable line with a neighboring fixed line.

where $X^{\otimes [j]}$ is either $X^{\otimes j}$ or $X^{\otimes j'} \otimes Y \otimes X^{\otimes j''}$ depending on whether $[j]$ is an integer or a pair of integers. Although all spaces in (2.2) are linearly isomorphic to $X^{\otimes n} \otimes Y^{\otimes m}$, the

brackets must not be removed in this type of expressions; the bracket positions actually characterize the underlying cell.

To define the local system $\mathcal{L}_{m,n}$, it remains to specify the morphisms between vector spaces of form (2.2) for the mergers described in **2.1.2** — the restriction maps $\Gamma(F_a^k) \rightarrow \Gamma(F_b^{k-1})$ corresponding to each case where F_b^{k-1} is in the closure of F_a^k . These maps can be regarded as a procedure to “collectivize” the crosses populating these lines.

- (1) In the case of a merger of two movable lines that originally carried ℓ and j crosses, the crosses can be “collectivized” in $\binom{\ell+j}{\ell}$ different ways, labeled by shuffle permutations $\sigma \in \mathbb{S}_{\ell,j}^{\ell+j} \subset \mathbb{S}_{\ell+j}$. For each such σ , let $\hat{\sigma}$ be its Matsumoto section (see **A.2**),

$$\hat{\sigma} : \Gamma(F_a^k) \rightarrow \Gamma(F_b^{k-1})$$

Here, $\Gamma(F_a^k)$ is a product (2.2), necessarily of the form $\dots (X^{\otimes \ell}) \otimes (X^{\otimes j}) \dots$; the map $\hat{\sigma}$ is nontrivial only on these two factors, and we do not write the rest of the tensor factors. Then the restriction map $\Gamma(F_a^k) \rightarrow \Gamma(F_b^{k-1})$ is given by the sum of the above $\hat{\sigma}$ over all shuffle permutations $\sigma \in \mathbb{S}_{\ell,j}^{\ell+j}$. By definition, this is the braided binomial $\mathbb{W}_{\ell,j}$ (see **A.2**):

$$\mathbb{W}_{\ell,j} \equiv \sum_{\sigma \in \mathbb{S}_{\ell,j}^{\ell+j}} \hat{\sigma} : (X^{\otimes \ell}) \otimes (X^{\otimes j}) \rightarrow (X^{\otimes (\ell+j)}).$$

- (2) In the case of a merger of a movable line carrying j crosses with a fixed line carrying ℓ' crosses on the left and ℓ'' crosses on the right of the puncture, any number b , $0 \leq b \leq j$, of the j crosses may go to the left of the puncture (again, we understand “left” and “right” in terms of the chosen orientation on the lines). This gives $j+1$ distinct F_b^{k-1} ; for each b , the number of possibilities of collectivizing the crosses is $\binom{b+\ell'}{b} \binom{j-b+\ell''}{j-b}$. The relevant map $\Gamma(F_a^k) \rightarrow \Gamma(F_b^{k-1})$ is different from the identity only in the component

$$\mu_{b;j,\ell',\ell''} : (X^{\otimes j}) \otimes (X^{\otimes \ell'} \otimes Y \otimes X^{\otimes \ell''}) \rightarrow (X^{\otimes (\ell'+b)} \otimes Y \otimes X^{\otimes (\ell''+j-b)}),$$

where it is given by (see Fig. 2.1)

$$(2.3) \quad \mu_{b;j,\ell',\ell''} = \mathbb{W}_{b,\ell'} \circ \mathbb{W}_{j-b,\ell''}^{\uparrow(b+\ell'+1)} \circ \Psi_{j-b,\ell'+1}^{\uparrow b}.$$

2.1.4. A useful *algebraic* simplification in the second case of line merger can be achieved by considering the sum of the $\mu_{b;j,\ell',\ell''}$ maps over all $0 \leq b \leq j$. The braided binomial

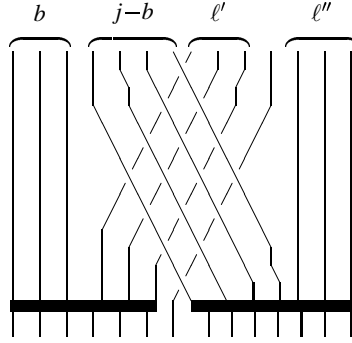


FIGURE 2.1. Illustration of formula (2.3). Read right to left (which corresponds to reading the diagram from top down), this formula tells the following: first, $j - b$ crosses (representing $(X^{\otimes(j-b)})$) “travel past” ℓ' crosses and the puncture (representing $X^{\otimes \ell'} \otimes Y$); next, the $j - b$ and ℓ'' are collectivized by taking a sum over $\binom{j-b+\ell''}{j-b}$ shuffle permutations, which yields $\mathbb{W}_{j-b, \ell''}$ (the right dark strip in the picture); and finally, the remaining b are similarly collectivized with the ℓ' crosses, the sum over the $\binom{b+\ell'}{b}$ shuffle permutations yielding the braided binomial $\mathbb{W}_{b, \ell'}$ (the left dark strip).

identity⁶

$$\sum_{b=0}^j \mu_{b; j, \ell', \ell''} = \mathbb{W}_{j, \ell' + 1 + \ell''}$$

then shows that the $j + 1$ cases of different collectivization types, taken together, are conveniently described by the map $\Gamma(F_a^k) \rightarrow \bigoplus_{b=0}^j \Gamma(F_b^{k-1})$ whose nontrivial component is

$$\mathbb{W}_{j, \ell' + \ell'' + 1} : (X^{\otimes j}) \otimes (X^{\otimes \ell'} \otimes Y \otimes X^{\otimes \ell''}) \rightarrow \bigoplus_{b=1}^j (X^{\otimes(\ell' + b)} \otimes Y \otimes X^{\otimes(\ell'' + j - b)}).$$

2.2. Homology complex. The above construction is neatly summarized if our local system \mathcal{L}_m is used, indeed, as a local system of coefficients. The homology complex of $\mathbb{X}_{m, n}$ with coefficients in \mathcal{L}_m is

$$(2.4) \quad 0 \leftarrow \mathcal{C}_{m, n} \xleftarrow{\partial} \mathcal{C}_{m, n+1} \xleftarrow{\partial} \dots \xleftarrow{\partial} \mathcal{C}_{m, 2n} \leftarrow 0,$$

⁶Which is a “quantum” counterpart of the identity

$$\sum_{b=0}^j \binom{b + \ell'}{b} \binom{j - b + \ell''}{j - b} = \binom{j + \ell' + \ell'' + 1}{j},$$

where the left-hand side is the total number of different possibilities of collectivizing the crosses for all $0 \leq b \leq j$, and the right-hand side is the number of shuffle permutations that “mix” j crosses with $\ell' + 1 + \ell''$ “points.”

where

$$\mathcal{C}_k = \bigoplus_{([j_1], [j_2], \dots, [j_{k-n}])} (X^{\otimes [j_1]}) \otimes (X^{\otimes [j_2]}) \otimes \dots \otimes (X^{\otimes [j_{k-n}]})$$

with the sum taken over all sets $([j_1], [j_2], \dots, [j_{k-n}])$ such that $\sum_{r=1}^{k-n} [j_r] = n$ and each set contains exactly m pairs (j'_i, j''_i) . The differential acts on each space (2.2) by the braid group algebra element

$$(2.5) \quad \partial = \sum_{i=2}^s (-1)^i \mathbb{W}_{[j_{i-1}], [j_i]}^{\uparrow(\#[j_1] + \dots + \#[j_{i-2}])}$$

where we define the “length”

$$(2.6) \quad \#[j] = \begin{cases} j, & [j] = j, \\ j' + j'' + 1, & [j] = (j', j'') \end{cases}$$

and set

$$(2.7) \quad \mathbb{W}_{[j], [\ell]} = \begin{cases} 0, & [j] = (j', j'') \text{ and } [\ell] = (\ell', \ell''), \\ \mathbb{W}_{\#[j], \#[\ell]}, & \text{otherwise,} \end{cases}$$

which map between the relevant spaces as follows:

$$(X^{\otimes [j]}) \otimes (X^{\otimes [\ell]}) \rightarrow \begin{cases} X^{\otimes (j+\ell)}, & [j] = j \text{ and } [\ell] = \ell, \\ \bigoplus_{a=0}^j X^{\otimes (\ell'+a)} \otimes Y \otimes X^{\otimes (j-a+\ell'')}, & [j] = j \text{ and } [\ell] = (\ell', \ell''), \\ \bigoplus_{a=0}^{\ell} X^{\otimes (j'+a)} \otimes Y \otimes X^{\otimes (\ell-a+j'')}, & [j] = (j', j'') \text{ and } [\ell] = \ell. \end{cases}$$

2.2.1. Example. We write explicitly how the differential acts on the space

$$(Y \otimes X) \otimes (X^2) \otimes (X \otimes Y) = \Gamma(F^5),$$

whose underlying cell is $F^5 = ((0, 1), 2, (1, 0))$. In codimension 1, the cell boundary is the union of cells $((0, 1), (3, 0)) \cup ((0, 3), (1, 0)) \cup ((0, 1), (2, 1)) \cup ((1, 2), (1, 0)) \cup ((0, 1), (1, 2)) \cup ((2, 1), (1, 0))$. Under each term in the differential, we indicate the space *to which* it maps:

$$\begin{aligned} \partial \big|_{(Y \otimes X) \otimes (X^2) \otimes (X \otimes Y)} = & - \frac{\text{id}}{(Y \otimes X) \otimes (X^3 \otimes Y)} + \frac{\text{id}}{(Y \otimes X^3) \otimes (X \otimes Y)} - \frac{\Psi_4}{(Y \otimes X) \otimes (X^3 \otimes Y)} + \frac{\Psi_2}{(Y \otimes X^3) \otimes (X \otimes Y)} \\ & - \frac{\Psi_3 \Psi_4}{(Y \otimes X) \otimes (X^3 \otimes Y)} + \frac{\Psi_3 \Psi_2}{(Y \otimes X^3) \otimes (X \otimes Y)} - \frac{\Psi_5 \Psi_4}{(Y \otimes X) \otimes (X^2 \otimes Y \otimes X)} + \frac{\Psi_1 \Psi_2}{(X \otimes Y \otimes X^2) \otimes (X \otimes Y)} \\ & - \frac{\Psi_3 \Psi_5 \Psi_4}{(Y \otimes X) \otimes (X^2 \otimes Y \otimes X)} + \frac{\Psi_1 \Psi_3 \Psi_2}{(X \otimes Y \otimes X^2) \otimes (X \otimes Y)} - \frac{\Psi_4 \Psi_3 \Psi_5 \Psi_4}{(Y \otimes X) \otimes (X \otimes Y \otimes X^2)} + \frac{\Psi_2 \Psi_1 \Psi_3 \Psi_2}{(X^2 \otimes Y \otimes X) \otimes (X \otimes Y)} \end{aligned}$$

(the two identity operators are different operators because they map to different spaces).

2.3. Flat bundle. The local system $\mathcal{L}_{m,n}$ can be alternatively described in terms of horizontal sections of a flat bundle over $\mathbb{X}_{m,n}$. The total space of the bundle is the quotient of

$$((\mathbb{C}_{w_1, \dots, w_m})^{\times n} \setminus \Delta) \times (X^{\otimes n} \otimes Y^{\otimes m})$$

by the identifications

$$s((\mathbb{C}_{w_1, \dots, w_m})^{\times n} \setminus \Delta) \times (X^{\otimes n} \otimes Y^{\otimes m}) \sim ((\mathbb{C}_{w_1, \dots, w_m})^{\times n} \setminus \Delta) \times \hat{s}(X^{\otimes n} \otimes Y^{\otimes m}),$$

where \hat{s} is the Matsumoto section of $s \in \mathbb{S}_n$, which is a linear operator defined on $X^{\otimes n} \otimes Y^{\otimes m}$.

For the nonpunctured complex plane, the flat bundle is of course equivalently specified by a morphism $\pi_1(\mathbb{X}_{0,n}) \rightarrow G$, where G is the structure group, $G \subset \text{End}(X^{\otimes n})$, and $\pi_1(\mathbb{X}_{0,n}) = \mathbb{B}_n$ —that is, by a braid group representation (the one we started with, of course).

2.4. Algebraic structures associated with the chains. The braided linear space

$$\mathcal{H}_+ = \text{---} \times \text{---} \oplus \text{---} \times \text{---} \times \text{---} \oplus \text{---} \times \text{---} \times \text{---} \times \text{---} \oplus \dots = \bigoplus_{j \geq 1} (X^{\otimes j}),$$

plays a special role in what follows. An associative product can be constructed on \mathcal{H}_+ as a composition of the tensor product and the differential, i.e., $x \cdot x' = \partial(x \otimes x')$. For $x \in (X^{\otimes j})$ and $x' \in (X^{\otimes j'})$, this gives

$$(2.8) \quad x \cdot x' = \partial(x \otimes x') = \coprod_{j, j'} (x \otimes x') \in (X^{\otimes (j+j')}).$$

By the same pattern, an $x \in (X^{\otimes j})$ acts on a $y \in X^{\otimes (j', j'')} = X^{\otimes j'} \otimes Y \otimes X^{\otimes j''}$ from the left and from the right as

$$(2.9) \quad x \cdot y = \partial(x \otimes y) = \coprod_{j, j' + j'' + 1} (x \otimes y), \quad y \cdot x = \partial(y \otimes x) = \coprod_{j' + j'' + 1, j} (y \otimes x).$$

2.4.1. Augmentation of \mathcal{H}_+ . We next let \mathcal{H} be the unital algebra obtained by augmenting (\mathcal{H}_+, \cdot) with a formal chain 1—the tensor unit in the monoidal category of braided linear spaces, which can be regarded as an object of type (2.2) with zero number of X or Y , i.e., as $(X^{\otimes 0})$. Then

$$(2.10) \quad \mathcal{H} = \mathcal{H}(X) = \bigoplus_{j \geq 0} (X^{\otimes j})$$

The action of differential (2.5) is immediately extended to any chain with any number of (\cdot) , by setting $\coprod_{0,n} = \coprod_{n,0} = \text{id}$. Hence, $(\cdot) \cdot x = x \cdot (\cdot) = x$ for any chain x .

2.4.2. Coproduct. The algebra $\mathcal{H}(X)$ can be made into a bialgebra by line cutting (or splitting, depending on the interpretation). For any line with n crosses, the coproduct is the result of cutting the line into two and summing over the possibilities of how many crosses occur to the left and to the right of the cut,

$$(2.11) \quad \Delta: \begin{array}{c} \text{---} \times \text{---} \times \text{---} \times \text{---} \end{array} \mapsto \begin{array}{c} \text{---} \times \times \times \text{---} \mid \text{---} \text{---} \text{---} \text{---} \end{array} + \begin{array}{c} \text{---} \times \times \text{---} \mid \text{---} \times \text{---} \text{---} \end{array} \\ + \begin{array}{c} \text{---} \text{---} \times \text{---} \mid \text{---} \times \times \text{---} \end{array} + \begin{array}{c} \text{---} \text{---} \text{---} \mid \text{---} \times \times \times \text{---} \end{array}$$

with the line cutting symbol \mid then to be read as \otimes ,

$$= (X^{\otimes 3}) \otimes () \oplus (X^{\otimes 2}) \otimes (X) \oplus (X) \otimes (X^{\otimes 2}) \oplus () \otimes (X^{\otimes 3})$$

(Figuratively speaking, “we do not know” where the cut happens to break the crosses—the *movable* points—into two groups, and “therefore” we sum over all possibilities.) This of course gives the well-known deconcatenation coproduct $\Delta: \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$,

$$(2.12) \quad \Delta: x_1 \otimes x_2 \otimes \dots \otimes x_n \mapsto \sum_{i=0}^n (x_1 \otimes \dots \otimes x_i) \otimes (x_{i+1} \otimes \dots \otimes x_n).$$

The above product and coproduct make $\mathcal{H}(X)$ into a braided bialgebra—which is in fact a braided Hopf algebra, our main subject in Sec. 3.

2.5. The legacy of screenings: a braided contour algebra. Here, mainly for illustrative purposes, we detail the correspondence between the “traditional” picture of screenings as contour integrals and the operations making $\mathcal{H}(X)$ in (2.10) into a braided Hopf algebra: in a particular example, we show that if elements of $\mathcal{H}(X)$ are represented in terms of contour integrals, then the Hopf algebra axioms indeed follow from certain rules for manipulating with these integrals.

We note that with the integral expressions in this subsection interpreted as elements in representation spaces of a braided Hopf algebra, the equality signs are typically to be understood as isomorphisms between different representations of the same object; “numerical” identities would then follow by taking matrix elements of the operators in relevant representations and explicitly inserting the isomorphisms (which we do not do here).

2.5.1. \mathcal{H} and its tensor powers. We fix a basis F_j in the braided space X ,⁷ and construct a basis in \mathcal{H} , $(F_{j_1}, \dots, F_{j_r}) \in X^{\otimes r}$, as the integrals

$$(2.13) \quad (F_{j_1}, \dots, F_{j_r}) = \int \cdots \int_{-\infty < z_1 < \cdots < z_r < \infty} f_{j_1}(z_1) \cdots f_{j_r}(z_r),$$

⁷In actual CFT examples, of course, the logic is just the reverse: X is the span of elements of a preferred basis, that of screenings.

where, by definition, the $f_j(z)$ (with the arguments ordered along a line) satisfy the exchange relations

$$(2.14) \quad f_i(u)f_j(z) = \Psi_{i,j}^{k,l} f_k(z)f_l(u), \quad u > z.$$

Elements of $\mathcal{H} \otimes \mathcal{H}$ are, by definition, two objects of form (2.13) “placed next to each other.” A useful convention is to represent them in the form with the two lines joined into a single one through an intermediate point a :

$$(2.15) \quad \mathcal{H} \otimes \mathcal{H} \ni (F_{i_1}, \dots, F_{i_r} \mid F_{j_1}, \dots, F_{j_s}) = \int \cdots \int_{-\infty < z_1 < \cdots < z_r < a < u_1 < \cdots < u_s < \infty} f_{i_1}(z_1) \cdots f_{i_r}(z_r) f_{j_1}(u_1) \cdots f_{j_s}(u_s).$$

All such integrals form a basis of $\mathcal{H} \otimes \mathcal{H}$. The point a is fixed here, and expressions (2.15) with different a are isomorphic representations of the same $\mathcal{H} \otimes \mathcal{H}$ element (identifying (2.15) with the same object where a is replaced with b means identifying two equivalent representations of the same algebraic object).

This generalizes to multiple tensor products; for example, a basis in $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$ is given by

$$(F_{i_1}, \dots, F_{i_r} \mid F_{j_1}, \dots, F_{j_s} \mid F_{k_1}, \dots, F_{k_p}) = \int \cdots \int_{-\infty < z_1 < \cdots < z_r < a < u_1 < \cdots < u_s < b < v_1 < \cdots < v_p < \infty} f_{i_1}(z_1) \cdots f_{i_r}(z_r) f_{j_1}(u_1) \cdots f_{j_s}(u_s) f_{k_1}(v_1) \cdots f_{k_p}(v_p).$$

2.5.2. Multiplication of the integrals. We now reformulate multiplication (2.8) in terms of the above integrals, i.e., express $m : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H}$ in the bases (2.15) and (2.13). The multiplication is the map in

$$m : (F_{i_1}, \dots, F_{i_r} \mid F_{j_1}, \dots, F_{j_s}) \mapsto \int \cdots \int_{-\infty < z_1 < \cdots < z_r < \infty} f_{i_1}(z_1) \cdots f_{i_r}(z_r) \int \cdots \int_{-\infty < u_1 < \cdots < u_s < \infty} f_{j_1}(u_1) \cdots f_{j_s}(u_s),$$

where braiding relations (2.14) are to be used to express the result in terms of basis elements (2.13). In the simplest case with $r = s = 1$, this becomes

$$m : (F_i \mid F_j) \mapsto \int_{-\infty}^{\infty} f_i(z) \int_{-\infty}^{\infty} f_j(u) = \iint_{-\infty < z < u < \infty} f_i(z) f_j(u) + \iint_{-\infty < z < u < \infty} f_i(u) f_j(z)$$

which we rewrite using the braiding as

$$\begin{aligned} &= \iint_{-\infty < z < u < \infty} f_i(z) f_j(u) + \Psi_{i,j}^{k,l} \iint_{-\infty < z < u < \infty} f_k(z) f_l(u) = (\delta_i^k \delta_j^l + \Psi_{i,j}^{k,l})(F_k, F_l) \\ &= (\mathbb{W}_{1,1})_{i,j}^{k,l}(F_k, F_l), \end{aligned}$$

where $\text{id} + \Psi = \mathbb{M}_{1,1}$ is the simplest quantum shuffle (see **A.4**). In general,

$$(2.16) \quad m : (F_{i_1}, \dots, F_{i_r} \mid F_{j_1}, \dots, F_{j_s}) \mapsto (\mathbb{M}_{r,s})_{i_1, \dots, i_r, j_1, \dots, j_s}^{k_1, \dots, k_{r+s}} (F_{k_1}, \dots, F_{k_{r+s}}),$$

which reproduces the product (2.8) in the bases given in (2.15) and (2.13).

2.5.3. Comultiplication of the integrals. The coproduct $\Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ defined in **2.4.2** can be conveniently represented as in (2.11), i.e., by inserting a cut into the integration domain in (2.13). The simplest case is merely

$$\Delta : \int_{-\infty}^{\infty} f_i(z) \mapsto \int_{-\infty}^a f_i(z) + \int_a^{\infty} f_i(z).$$

As in **2.4.1**, where the void was interpreted as the unit for the tensor product, working now in terms of bases we interpret an empty line, semi-infinite interval, or finite interval as the algebra unit. This means that $\int_{-\infty}^a f_i(z)$ is identified with $\times - - | - - -$ and consequently with $F_i \otimes 1$. Hence, $\Delta F_i = F_i \otimes 1 + 1 \otimes F_i$. In general, cutting the line carrying $(F_{i_1}, \dots, F_{i_r})$ at a point a yields $r + 1$ cases $z_1 < \dots < z_r < a$, $z_1 < \dots < z_{r-1} < a < z_r$, \dots , $a < z_1 < \dots < z_r$, and hence the deconcatenation coproduct

$$(2.17) \quad \Delta : (F_{i_1}, \dots, F_{i_r}) \mapsto (F_{i_1}, \dots, F_{i_r}) \otimes 1 \\ + (F_{i_1}, \dots, F_{i_{r-1}}) \otimes (F_{i_r}) + (F_{i_1}, \dots, F_{i_{r-2}}) \otimes (F_{i_{r-1}}, F_{i_r}) + \dots + 1 \otimes (F_{i_1}, \dots, F_{i_{r-1}}).$$

2.5.4. The antipode: contour reversal. The antipode map on integrals is naturally given by contour reversal. Applied to (2.13), this operation yields

$$\mathcal{S}((F_{j_1}, \dots, F_{j_r})) = (-1)^r \int_{-\infty < z_r < \dots < z_1 < \infty} \dots \int f_{j_1}(z_1) \dots f_{j_r}(z_r).$$

Using the braiding relations, we then reorder the f_{j_k} such that they occur in the same order as the integration contours. This is achieved by the product of braid group generators $\Psi_1(\Psi_2\Psi_1)(\Psi_3\Psi_2\Psi_1)\dots(\Psi_{r-1}\Psi_{r-2}\dots\Psi_1)$, a Matsumoto lift of the longest element of the symmetric group.

2.5.5. Verifying Hopf algebra axioms: an example. We now show by rearranging the integrals that (a very particular instance of) the braided bialgebra axiom (B.1) holds for product (2.16) and coproduct (2.17). The purpose is to illustrate the fact that the braided Hopf algebra $\mathcal{H}(X)$ introduced in **2.4** is a natural formalization of contour manipulation rules.

We evaluate both sides of (B.1) on the element

$$(2.18) \quad (F_i \mid F_j) = \int_{-\infty}^a f_i(z) \int_a^{\infty} f_j(w) \in \mathcal{H} \otimes \mathcal{H},$$

The left-hand side of (B.1) then becomes

$$(2.19) \quad \Delta \circ m((F_i \mid F_j)) =$$

$$(\delta_i^k \delta_j^l + \Psi_{i,j}^{k,l}) \left(\int_{-\infty < z < w < a} f_k(z) f_l(w) + \int_{-\infty < z < a < w < \infty} f_k(z) f_l(w) + \int_{a < z < w < \infty} f_k(z) f_l(w) \right),$$

which is simply $(\delta_i^k \delta_j^l + \Psi_{i,j}^{k,l})((F_k | F_l) \otimes 1 + (F_k) \otimes (F_l) + 1 \otimes (F_k | F_l)) \in \mathcal{H} \otimes \mathcal{H}$.

The right-hand side of (B.1) is $(m \otimes m) \circ (\text{id} \otimes \Psi \otimes \text{id}) \circ (\Delta \otimes \Delta)$. Applying first the coproduct to each factor in (2.18) yields

$$(2.20) \quad (\Delta \otimes \Delta)(F_i | F_j) = \int_{-\infty}^{a_1} f_i(z) \int_a^{a_3} f_j(w) + \int_{-\infty}^{a_1} f_i(z) \int_{a_3}^{\infty} f_j(w) \\ + \int_{a_1}^a f_i(z) \int_a^{a_3} f_j(w) + \int_{a_1}^a f_i(z) \int_{a_3}^{\infty} f_j(w),$$

where the integrations in each term in are split by the three points $-\infty < a_1 < a < a_3 < \infty$, and hence each term is an element of $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$. Next, in applying $\text{id} \otimes \Psi \otimes \text{id}$ to (2.20), the braiding is nontrivial in only one term, and hence

$$(2.21) \quad (\text{id} \otimes \Psi \otimes \text{id}) \circ (\Delta \otimes \Delta)(F_i | F_j) \\ = \int_{-\infty}^{a_1} f_i(z) \int_{a_1}^a f_j(w) + \int_{-\infty}^{a_1} f_i(z) \int_{a_3}^{\infty} f_j(w) + \Psi_{i,j}^{k,l} \int_{a_1}^a f_k(z) \int_a^{a_3} f_l(w) + \int_{a_1}^a f_i(z) \int_{a_3}^{\infty} f_j(w).$$

It remains to apply $m \otimes m$ to (2.21), which amounts to taking $a_1 \rightarrow -\infty$ and $a_3 \rightarrow a$ in all the lower integration limits, and $a_1 \rightarrow a$ and $a_3 \rightarrow +\infty$ in all the upper limits. Finally,

$$(m \otimes m) \circ (\text{id} \otimes \Psi \otimes \text{id}) \circ (\Delta \otimes \Delta)(F_i | F_j) \\ = \int_{-\infty}^a f_i(z) \int_{-\infty}^a f_j(w) + \int_{-\infty}^a f_i(z) \int_a^{\infty} f_j(w) \\ + \Psi_{i,j}^{k,l} \int_{-\infty}^a f_k(z) \int_a^{+\infty} f_l(w) + \int_a^{+\infty} f_i(z) \int_a^{\infty} f_j(w),$$

which coincides with (2.19) after the use of (two) formulas of the type $\int_{-\infty}^a f_i(z) \int_{-\infty}^a f_j(w) = (\delta_i^k \delta_j^l + \Psi_{i,j}^{k,l}) \iint_{-\infty < z < w < a} f_i(z) f_j(w)$.

3. $\mathcal{H}(X)$ AND $\mathfrak{B}(X)$ MODULES

In this section, we study (Hopf and Yetter–Drinfeld) modules of the braided Hopf algebra $\mathcal{H}(X)$ in (2.10), the shuffle algebra of a braided vector space X (whose copies were associated with screenings/crosses in 2.1). Constructing $\mathcal{H}(X)$ -modules requires the other braided vector space, Y , that we fixed in the beginning of Sec. 2. The techniques in this section generalize to the case where X and Y are objects of a braided monoidal category (\mathcal{C}, Ψ) (with mild additional assumptions whenever necessary), but we avoid extra complications by assuming the category to be that of braided linear spaces. The identities derived below for operators acting on the representations in fact require nothing

in addition to braiding, and can therefore be considered “universal,” i.e., holding in the braid group algebra \mathbb{B}_N with a sufficiently large N .

All constructions for $\mathcal{H}(X)$ in **3.1–3.4** restrict to the Nichols algebra $\mathfrak{B}(X)$.

3.1. The braided Hopf algebra $\mathcal{H}(X)$. The braided bialgebra $\mathcal{H} = \mathcal{H}(X) = T^\bullet(X)$ with the shuffle product (2.8) and deconcatenation coproduct (2.12) becomes a braided Hopf algebra with the antipode $\mathcal{S}_r : X^{\otimes r} \rightarrow X^{\otimes r}$ given by a signed the “half-twist”—the braid group element obtained via the Matsumoto lift of the longest element in the symmetric group:

$$(3.1) \quad \mathcal{S}_r = (-1)^r \Psi_1(\Psi_2 \Psi_1)(\Psi_3 \Psi_2 \Psi_1) \dots (\Psi_{r-1} \Psi_{r-2} \dots \Psi_1)$$

(with the brackets inserted to highlight the structure). For example,

$$\mathcal{S}_5 = - \left(\begin{array}{c} \diagdown \quad \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \quad \diagdown \end{array} \right)$$

Also, $\mathcal{S}_1 = -\text{id}$. The sign can be thought to follow from contour reversal when the elements of $\mathcal{H}(X)$ are viewed as multiple integrals of the screenings, as in Sec. 2. The counit is $\varepsilon : X^{\otimes r} \rightarrow \delta_{r,0}k$ (identity on k).

We note that Eq. (3.1) defines the action of \mathcal{S}_r on any tensor product of r objects from \mathcal{C} . With this understanding, we also define the *full twist* as the operator acting on any r -fold tensor product in the category,

$$(3.2) \quad \theta_r = \mathcal{S}_r \mathcal{S}_r.$$

Notation. With X fixed in what follows, we use the short notation $(r) = X^{\otimes r}$.

3.2. Hopf bimodules. We introduce a category of $\mathcal{H}(X)$ -modules, which turn out to also be comodules, and in fact Hopf bimodules.

With our second braided linear space Y , we consider the space

$$\begin{aligned} ((Y)) &= \bigoplus_{s,t \geq 0} X^{\otimes s} \otimes Y \otimes X^{\otimes t} \\ &\equiv \bigoplus_{s,t \geq 0} (s; Y; t), \end{aligned}$$

and make it into an $\mathcal{H}(X)$ -module bimodule as in (2.9). In our current notation, the left action $\mathcal{H}(X) \otimes ((Y)) \rightarrow ((Y))$ restricted to $(r) \otimes (s; Y; t)$ is the map

$$\sqcup_{r,s+1+t} : (r) \otimes (s; Y; t) \rightarrow \bigoplus_{i=0}^r (s+i; Y; t+r-i).$$

We often use the “infix notation” $(r) \cdot (s; Y; t)$ for it, and write

$$(3.3) \quad (r) \cdot (s; Y; t) = \sqcup_{r, s+1+t} ((r) \otimes (s; Y; t)) \subset \bigoplus_{i=0}^r (s+i; Y; t+r-i).$$

This map is illustrated in Fig. 1.2 in the case where $r = 2$, $s = 1$, and $t = 1$. In each term in the figure, the “incoming” strands (at the top) then correspond to X, X, X, Y , and X .

The right $\mathcal{H}(X)$ -action on $((Y))$, also in accordance with (2.9), is

$$(3.4) \quad (s; Y; t) \cdot (r) = \sqcup_{s+1+t, r} ((s; Y; t) \otimes (r)) \subset \bigoplus_{i=0}^r (s+i; Y; t+r-i).$$

Moreover, $((Y))$ is also a bicomodule via left and right deconcatenations:

$$\delta_L(s; Y; t) = \bigoplus_{i=0}^s (i) \otimes (s-i; Y; t), \quad \delta_R(r; Y; s) = \bigoplus_{i=0}^s (r; Y; s-i) \otimes (i).$$

The following fact is well known.

3.2.1. Proposition. *With the above $\mathcal{H}(X)$ action and coaction, $((Y))$ is a Hopf bimodule.*

We note that the *action* property for both left and right actions is expressed by one of the “basic” quantum shuffle identities (A.4). Some other identities expressing the statement in 3.2.1 are given in A.6.2.

3.2.2. Proposition. *On each graded component $(s; Y; t)$ of $((Y))$, the “relative antipode” defined in (B.5) acts as*

$$\sigma|_{(s; Y; t)} \equiv \sum_{i=0}^s \sum_{j=0}^t \sqcup_{s+1+t-j, j} \sqcup_{i, s-i+1+t-j} \mathcal{S}_i \mathcal{S}_j^{\uparrow(s+1+t-j)} = -\mathcal{S}_{s+1+t}.$$

We emphasize that the right-hand side, defined in (3.1), acts here by a flip of all the $(s+1+t)$ strands representing $X^{\otimes s} \otimes Y \otimes X^{\otimes t}$, $\mathcal{S}_{s+1+t} : (s; Y; t) \rightarrow (t; Y, s)$.

3.3. Yetter–Drinfeld modules and the adjoint action. We use the Hopf bimodules discussed above to construct (left–left) Yetter–Drinfeld $\mathcal{H}(X)$ modules. In any braided monoidal category (with split idempotents), the space of right coinvariants of a Hopf bimodule is a left–left Yetter–Drinfeld module under the left adjoint action (see [67, 68] and the references therein). The relevant definitions are recalled in (B.7) and (B.6). We now translate the adjoint action defined in (B.6) into the shuffle language. The main result is in 3.3.3 below.

3.3.1. Rewriting (B.6) in terms of shuffles. On all of $((Y))$, the adjoint action (B.6) readily reformulates as the map $\blacktriangleright : \mathcal{H}(X) \otimes ((Y)) \rightarrow ((Y))$ whose restriction

$$\blacktriangleright : (r) \otimes (s; Y; t) \rightarrow \bigoplus_{i=0}^r (s+i; Y; t+r-i)$$

is (again, in the “infix notation”) given by

$$(3.5) \quad (r) \blacktriangleright (s; Y; t) = \sum_{i=0}^r \underbrace{\mathbb{W}_{r-i, s+1+t+i}}_{\cdot} \underbrace{\mathbb{W}_{s+1+t, i}^{\uparrow(r-i)}}_{\cdot} \underbrace{\mathcal{S}_i^{\uparrow(r-i+s+1+t)}}_S \underbrace{\Psi_{i, s+1+t}^{\uparrow(r-i)}}_{\Psi} \underbrace{\Delta}_{\Delta} ((r) \otimes (s; Y; t)).$$

For clarity of comparison with (B.6), we placed the ingredients of that formula under the corresponding pieces of the “shuffle” formula; the deconcatenation coproduct Δ can be conventionally localized at the first (counting from the right) occurrence of the index i .

3.3.2. Right coinvariants in the Yetter–Drinfeld module $((Y|$. With the right coaction given also by deconcatenation, the space of right coinvariants in $((Y))$ is simply

$$((Y| \equiv \bigoplus_{s \geq 0} (s; Y;) = \bigoplus_{s \geq 0} (s; Y),$$

where we also somewhat streamlined the notation by omitting a redundant semicolon separating the void for $t = 0$. We now describe a nicer formula for $(r) \blacktriangleright (s; Y)$ than just (3.5) with $t = 0$.

Let \curvearrowright be the left action $\mathcal{H}(X) \otimes ((Y| \rightarrow ((Y|$:

$$\tilde{\mathcal{L}}_{r,s} = \mathbb{W}_{r,s} : (r) \otimes (s; Y) \rightarrow (r+s; Y)$$

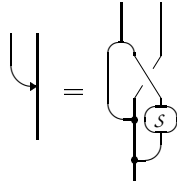
and \curvearrowleft be the right action $((Y| \otimes \mathcal{H}(X) \rightarrow ((Y|$:

$$\tilde{\mathcal{R}}_{s,r} = \mathbb{W}_{s,r} \Psi_{1,r}^{\uparrow s} : (s; Y) \otimes (r) \rightarrow (s+r; Y).$$

These left and right actions commute with each other, and *restrict to the spaces of right coinvariants* $((Y| \subset ((Y))$ (which we have just used by specifying the targets as $((Y|$, not $((Y))$). We also note for the future use that $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{R}}$ respectively satisfy the left–left and left–right Hopf module axioms

$$(3.6) \quad \begin{array}{c} \curvearrowright \\ \text{---} \end{array} = \begin{array}{c} | \\ \text{---} \end{array} \quad \text{and} \quad \begin{array}{c} \curvearrowleft \\ \text{---} \end{array} = \begin{array}{c} | \\ \text{---} \end{array}$$

3.3.3. Proposition. *The left adjoint action of $\mathcal{H}(X)$ on $((Y|$ restricted to $(r) \otimes (s; Y)$ is the map $(r) \otimes (s; Y) \rightarrow (s + r; Y)$ given by*



in terms of the \frown and \smile actions, that is,

$$\begin{aligned}
 (3.7) \quad (r) \blacktriangleright (s; Y) &= \sum_{i=0}^r \mathbb{W}_{r-i, s+i} \mathbb{W}_{s,i}^{\uparrow(r-i)} (\mathcal{S}_i \Psi_{1,i} \Psi_{i,1})^{\uparrow(r-i+s)} \Psi_{i,s}^{\uparrow(r-i)} (s+r; Y) \\
 &= \sum_{i=0}^r \mathbb{W}_{r-i, s+i} (\mathbb{W}_{s,i} \Psi_{1,i}^{\uparrow s} \mathcal{S}_i^{\uparrow(s+1)} \Psi_{i, s+1})^{\uparrow(r-i)} (s+r; Y).
 \end{aligned}$$

The structure of (3.7) is already seen from the lowest cases:

$$(3.8) \quad (1) \blacktriangleright (s; Y) = (\mathbb{W}_{1,s} - \mathbb{W}_{s,1} \Psi_{s+1} \Psi_{s+1} \Psi_s \dots \Psi_1) (s+1; Y),$$

$$\begin{aligned}
 (3.9) \quad (2) \blacktriangleright (s; Y) &= (\mathbb{W}_{2,s} - \mathbb{W}_{1,s+1} \mathbb{W}_{s,1}^{\uparrow 1} \Psi_{s+2} \Psi_{s+2} \dots \Psi_2 \\
 &\quad + \mathbb{W}_{s,2} \Psi_{s+1} \Psi_{s+2} (\Psi_{s+1} \Psi_{s+1} \dots \Psi_1) \Psi_{s+2} \dots \Psi_2) (s+2; Y).
 \end{aligned}$$

The right-hand side of (3.8) expands into $2s+2$ terms illustrated in Fig. 1.4 for $s=2$, when the two braided integers $\mathbb{W}_{1,s}$ and $\mathbb{W}_{s,1}$ become $\mathbb{W}_{1,2} = \text{id} + \Psi_1 + \Psi_2 \Psi_1$ and $\mathbb{W}_{2,1} = \text{id} + \Psi_2 + \Psi_1 \Psi_2$. The leftmost strand representing the space $(1) = X$ in Fig. 1.4 braids with each of the X spaces in $(s) = X^{\otimes s}$ one by one (terms with the plus sign, all following from the expansion of $\mathbb{W}_{1,s}$), or braids with the last, $(s+1)$ th strand representing Y , but does not stay to the right of it; instead, it is immediately braided with it again and, “on its way back,” is braided with each of the s strands (the “minus” terms). The right-hand side of (3.9) expands similarly but more interestingly, as is illustrated in Fig. 3.1 for $s=1$.

3.3.4. Proof of (3.7). The proof is in fact elementary. From (3.5), we know that the left adjoint action on $(s; Y)$ is

$$(r) \blacktriangleright (s; Y) = \sum_{i=0}^r \mathbb{W}_{r-i, s+1+i} \mathbb{W}_{s+1,i}^{\uparrow(r-i)} \mathcal{S}_i^{\uparrow(r-i+s+1)} \Psi_{i, s+1}^{\uparrow(r-i)} ((r) \otimes (s; Y)).$$

The claim is therefore equivalent to the identity

$$\begin{aligned}
 \sum_{i=0}^r \mathbb{W}_{r-i, s+1+i} \mathbb{W}_{s+1,i}^{\uparrow(r-i)} \mathcal{S}_i^{\uparrow(r-i+s+1)} \Psi_{i, s+1}^{\uparrow(r-i)} \\
 = \sum_{i=0}^r \mathbb{W}_{r-i, s+i} (\mathbb{W}_{s,i} \Psi_{1,i}^{\uparrow s} \mathcal{S}_i^{\uparrow(s+1)} \Psi_{i, s+1})^{\uparrow(r-i)}.
 \end{aligned}$$

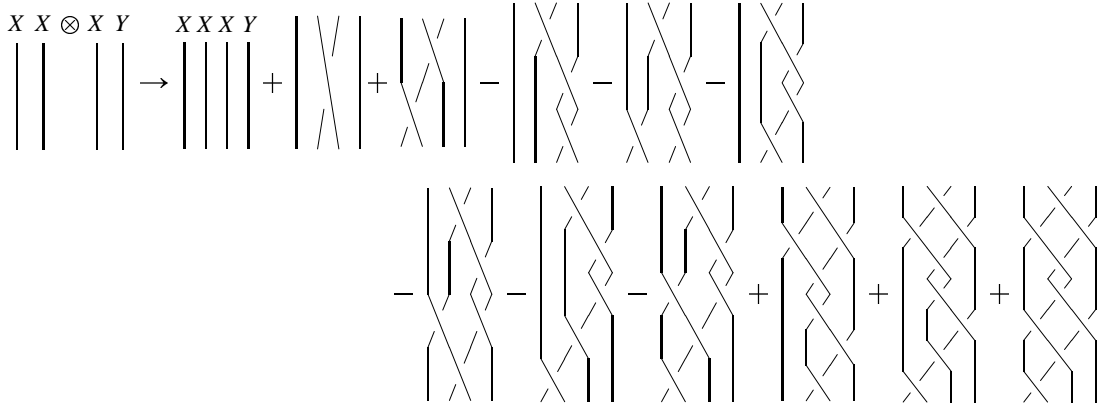


FIGURE 3.1. Adjoint action $(r) \blacktriangleright (s; Y)$ for $r = 2$ and $s = 1$. The “XY” assignment shown in the first term in the right-hand side is the same in all terms. The first three terms on the right are the expansion of the first term in (3.9) and represent the shuffling of the two “new” strands with the s X -strands in $(s; Y)$; the last, Y strand then remains “passive.” The term with the minus sign in (3.9) expands into the “minus” terms in the figure, where the *second* of the new strands winds around the Y strand and returns (by the leftmost Ψ_{s+2}), to be shuffled (by $\sqcup_{s,1}^{\uparrow 1}$) with the s X -strands; the resulting $s + 1$ strands are then shuffled (by $\sqcup_{1,s+1}$) with the *first* of the new strands. In the last term in (3.9), both new strands wind around the last (Y -)strand: the second goes to the right of Y via $\Psi_{s+2} \dots \Psi_1$ and “waits” there for the first to wind around Y and return (the inner brackets), and then returns itself ($\Psi_{s+1} \Psi_{s+2}$), after which the two are shuffled by $\sqcup_{s,2}$ with the s original X -strands. (That the two “traveler” strands swap their order in the final position reflects the presence of the antipode in (3.7).)

But $\sqcup_{s,i} \Psi_{1,i}^{\uparrow s} = \sqcup_{s+1,i} - \sqcup_{s+1,i-1}$ by virtue of (A.2), and we can therefore continue as

$$= \sqcup_{r,s} + \sum_{i=1}^r \sqcup_{r-i,s+i} (\sqcup_{s+1,i}^{\uparrow(r-i)} - \sqcup_{s+1,i-1}^{\uparrow(r-i)}) \mathcal{S}_i^{\uparrow(r-i+s+1)} \Psi_{i,s+1}^{\uparrow(r-i)}.$$

Similar terms in the left- and the right-hand side are now combined using (A.2) once again, $\sqcup_{r-i,s+1+i} - \sqcup_{r-i,s+i} = \sqcup_{r-i-1,s+1+i} \Psi_{1,s+1+i}^{\uparrow(r-i-1)}$, and hence the claim equivalently reformulates as

$$\begin{aligned} \sqcup_{r,s+1} + \sum_{i=1}^{r-1} \sqcup_{r-i-1,s+1+i} \Psi_{1,s+1+i}^{\uparrow(r-i-1)} \sqcup_{s+1,i}^{\uparrow(r-i)} \mathcal{S}_i^{\uparrow(r-i+s+1)} \Psi_{i,s+1}^{\uparrow(r-i)} \\ = \sqcup_{r,s} - \sum_{i=1}^r \sqcup_{r-i,s+i} (\sqcup_{s+1,i-1} \mathcal{S}_i^{\uparrow(s+1)} \Psi_{i,s+1})^{\uparrow(r-i)}. \end{aligned}$$

The two isolated \sqcup -terms combine with the sum in the left-hand side, extending it to $i = 0$; we then shift the summation index to the range from 1 to r , as in the right-hand side. The resulting equality is a term-by-term identity. \square

3.3.5. In what follows, we use the notation $\text{Ad}_{r,s} : (r) \otimes (s; Y) \rightarrow (r+s; Y)$ for the “adjoint action operator” in (3.7):

$$\text{Ad}_{r,s}(r+s; Y) = (r) \blacktriangleright (s; Y).$$

The fact that the adjoint action is an *action* is expressed as

$$\text{Ad}_{r,s+t} \text{Ad}_{s,t}^{\uparrow r} = \text{Ad}_{r+s,t} \sqcup_{r,s}$$

for all $r, s, t \geq 0$.

3.3.6. Remark. We make contact with [11], where a formula for the left adjoint action was derived based on the operator $T_r : (r; Y) \rightarrow (r; Y)$ given by

$$T_r = (\text{id} - \Psi_r \Psi_r \Psi_{r-1} \dots \Psi_1) \dots (\text{id} - \Psi_r \Psi_r \Psi_{r-1})(\text{id} - \Psi_r \Psi_r), \quad r \geq 1,$$

and $T_0 = \text{id}$. This operator is related to the adjoint-action operator introduced above as

$$\mathfrak{S}_r T_r = \text{Ad}_{r,0} \mathfrak{S}_r,$$

with the “braided factorial” defined in **A.4**.

Examining the operators that act only on the last, Y strand in $(r) \blacktriangleright (s; Y)$, and then using induction on the number of strands counted from the right, we can restate **3.3.3** in a slightly more general form, which shows how the \blacktriangleright action on $(s+t; Y)$ “factors” (of course, via the coproduct) through \blacktriangleright acting on the *last* several strands, $(t; Y)$. The following proposition expresses a formula of the type $(r) \blacktriangleright (s+t; Y) = \sum \dots ((r') \blacktriangleright (t; Y)) \dots$

3.3.7. Proposition. *The left adjoint action satisfies the identity*

$$(3.10) \quad \begin{array}{c} (r) \quad (s+t; Y) \\ \downarrow \quad \downarrow \\ \text{[Diagram: A single strand from } (r) \text{ and a strand from } (s+t; Y) \text{ meeting at a point, with an arrow pointing to the right.]} \end{array} = \begin{array}{c} (r) \quad (s) \quad (t; Y) \\ \downarrow \quad \downarrow \quad \downarrow \\ \text{[Diagram: A complex braid-like structure involving strands from } (r), (s), \text{ and } (t; Y). \text{ It shows a series of crossings and a loop labeled } s \text{ on the rightmost strand.]} \end{array}$$

with the same left and right actions \mathfrak{U} and \mathfrak{V} as above.

3.3.8. Proposition (cf. [68]). *On each graded component $(s; Y)$ of $((Y|$, the “squared relative antipode” (B.10) acts as the full twist θ_{s+1} :*

$$\sigma_2|_{(s; Y)} \equiv \sum_{i=0}^s \text{Ad}_{i, s-i} S_i = \theta_{s+1}.$$

We emphasize that the full twist defined in (3.2) is here applied to all of the $s+1$ strands in $(s; Y)$; the relation nicely illustrates why σ_2 is a *squared* “antipode” in the terminology of [67, 68].

3.4. Multivertex Hopf bimodules. Hopf bimodules can be further constructed as $\mathcal{Y} \otimes \mathcal{Z}$, where \mathcal{Y} is a Yetter–Drinfeld module and \mathcal{Z} is a Hopf bimodule [31]. The tensor product is a Hopf bimodule under the right action and right coaction induced from those of \mathcal{Z} , and under the diagonal left action and codiagonal left coaction. For a Hopf bimodule $((Z))$ and a Yetter–Drinfeld module $((Y|$, the resulting Hopf bimodule has the graded components $(s; Y; t; Z; u)$.

Evidently, the construction can be iterated by further taking tensor products with Yetter–Drinfeld module(s); it is associative in the sense that the Yetter–Drinfeld modules can be multiplied first (to produce a Yetter–Drinfeld module under the diagonal action and codiagonal coaction). The resulting Hopf bimodules have the graded components

$$(3.11) \quad (s_1; Y_1; s_2; Y_2; \dots; s_n; Y_n; s_{n+1}) = X^{\otimes s_1} \otimes Y_1 \otimes X^{\otimes s_2} \otimes Y_2 \otimes \dots \otimes X^{\otimes s_n} \otimes Y_n \otimes X^{\otimes s_{n+1}}$$

(in our setting, we only encounter cases where $Z = Y_i = Y$, but it is useful to allow “different Y ” spaces in order to clarify the structure of formulas).

3.5. Introducing \mathcal{H}^* . We introduce a “conjugate” E to the adjoint action $F \equiv (1) \blacktriangleright$. Although this does not allow obtaining even a decent associative algebra in general, it is interesting to see that an “ $EF - FE$ ” formula can nevertheless be written with a meaningful right-hand side.

For $\mathcal{H}(X)$ in (2.10), we introduce the dual Hopf algebra \mathcal{A} built on the tensor algebra of $A = X^*$:

$$\begin{aligned} \mathcal{A} &= \bigoplus_{r \geq 0} A^{\otimes r}, \\ &= () \oplus (-1) \oplus (-2) \oplus \dots \end{aligned}$$

The pairing $\rho : A \otimes X \rightarrow k$ is denoted by \cup . It is extended to the tensor products as

$$\rho(A^{\otimes n}, X^{\otimes n}) = \left| \begin{array}{c} \dots \cup \dots \\ \cup \end{array} \right|$$

Then any left $\mathcal{H}(X)$ comodule becomes a left \mathcal{A} module under the action

$$\begin{array}{c} \mathcal{A} \\ \downarrow \\ \mathcal{H} \end{array} : \alpha \otimes h \mapsto \alpha \triangleright h = \rho(\alpha, h_{(-1)})h_{(0)}, \quad \alpha \in \mathcal{A}, \quad h \in \mathcal{H}.$$

With the deconcatenation coaction on $((Y))$, we then have the map $(-1) \otimes (s; Y) \rightarrow (s-1; Y)$ given by

$$(3.12) \quad \begin{array}{c} A \\ | \\ \otimes \end{array} \begin{array}{c} X \\ | \end{array} \begin{array}{c} X \\ | \end{array} \dots \begin{array}{c} X \\ | \end{array} \begin{array}{c} Y \\ | \end{array} \rightarrow \begin{array}{c} A \\ | \end{array} \begin{array}{c} X \\ | \end{array} \begin{array}{c} X \\ | \end{array} \dots \begin{array}{c} X \\ | \end{array} \begin{array}{c} Y \\ | \end{array}$$

(with s X -strands in the left-hand side and $s-1$ “free” ones in the right-hand side).

We introduce a new (and hopefully suggestive) notation for the maps in (3.12) and (3.8), writing them as

$$\begin{aligned} e_s &: (-1) \otimes (s; Y) \rightarrow (s-1; Y), \\ f_s &: (1) \otimes (s; Y) \rightarrow (s+1; Y), \end{aligned}$$

and “compare” the results of applying e and f in different orders. In considering the product $A \otimes X \otimes (s; Y) = (-1) \otimes (1) \otimes (s; Y)$, we *enumerate* the factors also as $-1, 1, 2, \dots, s+2$. Then, in particular, Ψ_1 (see **A.3**) acts here as $\text{id} \otimes \Psi \otimes \text{id}^{\otimes s}$, and the pairing between the A strand and the leftmost X strand is conveniently denoted as $\rho_{-1,1}$.

3.5.1. Lemma. *The following identity holds for maps $(-1) \otimes (1) \otimes (s; Y) \rightarrow \otimes(s; Y)$:*

$$e_{s+1} \circ f_s - f_{s-1} \circ e_s \circ \Psi_1 = \rho_{-1,1} \otimes \text{id}^{\otimes(s+1)} - \mathcal{K}_2(s+1),$$

where $\mathcal{K}_2(s+1) : (-1) \otimes (1) \otimes (s; Y) \rightarrow (s; Y)$ is the map

$$\mathcal{K}_2(s+1) = \rho_{-1,1} \circ (\Psi_1 \dots \Psi_{s+1} \Psi_{s+1} \dots \Psi_1) :$$

(the leftmost X strand is passed around the bunch of $s+1$ strands, and upon returning is contracted with the A strand).

The proof is straightforward. On $(-1) \otimes (1) \otimes (s; Y)$, we first apply e_s to $(s; Y)$, for which we need to contract (-1) with the first X strand in $(s; Y)$. For this, we first move

this strand to the left by Ψ_1 :

$$\rho_{-1,1} \circ \Psi_1 : \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array}$$

Next applying f_{s-1} gives the map

$$(\rho_{-1,1} \otimes f_{s-1}^{\uparrow 1}) \circ \Psi_1 = \rho_{-1,1} \circ (\text{id} \otimes f_{s-1}^{\uparrow 1}) \circ \Psi_1 : (-1) \otimes (1) \otimes (s; Y) \rightarrow (s; Y).$$

The first-f-then-e variant simply gives the map

$$\rho_{-1,1} \circ (\text{id} \otimes f_s) : (-1) \otimes (1) \otimes (s; Y) \rightarrow (s; Y).$$

The maps to the right of ρ in the last two formulas satisfy the identity $f_s - f_{s-1}^{\uparrow 1} \Psi_1 = \text{id} - \Psi_1 \dots \Psi_{s+1} \Psi_{s+1} \dots \Psi_1$, as is easily verified from the definition. The formula in the lemma now follows immediately.

In the case of *diagonal braiding*, the formula for \mathcal{K}_2 in the lemma reduces nicely, to the product of squared braiding with each of the modules in $(s; Y) = X^{\otimes s} \otimes Y$. We encounter a particular example in (5.28).

4. FUSION PRODUCT

Our aim in this section is to define a product of Yetter–Drinfeld modules of right coinvariants introduced in 3.3. As a linear space, this is the tensor product, but it carries the *adjoint* representation of $\mathcal{H}(X)$ (the “cumulative” adjoint, as we discuss in what follows). This construction plays a role in CFT applications, but may also be interesting in the “abstract” setting. Geometrically, in the spirit of Sec. 2, fusion corresponds to a degeneration of the picture as in Fig. 1.1, when several punctures are allowed to sit on the same line. On the algebraic side, fusion gives an ample supply of Yetter–Drinfeld $\mathcal{H}(X)$ and $\mathfrak{B}(X)$ modules.

4.1. Fusion product of Hopf bimodules. For Hopf bimodules described in the preceding section, their fusion product is the “X” map defined in (B.12). Its categorical meaning is that it projects onto the cotensor product. We now write (B.12) in terms of the braid group algebra generators (as a “quantum shuffle” formula). The governing idea is that for subspaces $(s_1; Y; s_2)$ and $(t_1; Z; t_2)$ of two Hopf bimodules (see 3.2), their product $(s_1; Y; s_2) \odot (t_1; Z; t_2)$ is given by “propagation” of the X -strands that “face the other module” to that module: the s_2 strands split as $(s_2 - i) + i$, and the t_1 , similarly, as $j + (t_1 - j)$; then (i) and (j) are braided with each other, after which (i) acts to the right, via (3.3) $(\sqcup_{i, t_1 - j + 1 + t_2})$, and (j) acts to the left, via (3.4) $(\sqcup_{s_1 + 1 + s_2 - i, j})$:

$$(4.1) \quad (s_1; Y; s_2) \odot (t_1; Z; t_2) =$$

$$(r) \otimes (s; Y; t; Z) \rightarrow \text{Ad}_{r,s+1+t}(r+s; Y; t; Z) \subset \bigoplus_{j=0}^r (s+j; Y; t+r-j; Z).$$

These statements are the result of a calculation showing that the above coaction and action are intertwined by \odot with the standard coaction and action on tensor products of Yetter–Drinfeld modules:

$$(4.4) \quad \begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array} \quad \text{and} \quad \begin{array}{c} (r)(s;Y)(t;Z) \\ \text{Diagram 3} \end{array} = \begin{array}{c} \text{Diagram 4} \end{array}$$

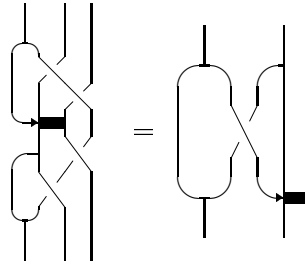
where $\begin{array}{c} \text{Diagram 5} \end{array}$ denotes the “cumulative” adjoint action $(r) \triangleright (s; Y; t; Z) = \text{Ad}_{r,s+1+t}(r+s; Y; t; Z)$.

The first intertwining identity in (4.4), for coactions, follows by noticing that it is a “dotted” version of (B.15) (for the right action $\tilde{\mathcal{R}} = \begin{array}{c} \text{Diagram 6} \end{array}$ from 3.3.2), and is proved the same *because of the second property in (3.6) for $\tilde{\mathcal{R}}$* ; just this property underlies the identity. To prove the second identity in (4.4), we combine 3.3.3 with the calculations in B.3 to obtain the chain of identities

$$\begin{array}{c} \text{Diagram 7} \end{array} \stackrel{3.3.3}{=} \begin{array}{c} \text{Diagram 8} \end{array} \stackrel{(B.17)}{=} \begin{array}{c} \text{Diagram 9} \end{array} = \begin{array}{c} \text{Diagram 10} \end{array} \stackrel{3.3.7}{=} \begin{array}{c} \text{Diagram 11} \end{array}$$

Here, invoking (B.17) refers to its “dotted” version (with $\tilde{\mathcal{R}}$ from 3.3.2 for all right actions and with $\tilde{\mathcal{L}}$ for all left actions except the adjoint), whose proof is identical to the proof given in Fig. B.1, again because properties (3.6) hold (both are needed this time). The next, unlabeled equality in the above chain once again uses 3.3.3 to recover the adjoint action.

4.2.2. Two-vertex Yetter–Drinfeld modules. *The Yetter–Drinfeld axiom holds for the coaction and action in 4.2.1,*

(4.5) 

simply because both diagram identities in (4.4) are *intertwining* formulas. We thus have *two-vertex* Yetter–Drinfeld modules with graded components $(s; Z; t; Y) = X^{\otimes s} \otimes Z \otimes X^{\otimes t} \otimes Y$.

4.2.3. Multivertex Yetter–Drinfeld modules. More Yetter–Drinfeld modules can be constructed via iterated fusion product, using the associativity of the \mathcal{U} map (see **B.3**). For example, in evaluating the product $((Z| \odot (((Y| \odot ((U|$), both $((Z|$ and $((Y| \odot ((U|$ are Yetter–Drinfeld modules under the left adjoint action; the construction is iterated smoothly to yield Yetter–Drinfeld modules with graded components $(s_1; Y_1, s_2; Y_2; \dots; s_n; Y_n)$ carrying the “cumulative” adjoint action

$$(r) \triangleright (s_1; Y_1, s_2; Y_2; \dots; s_n; Y_n) = \text{Ad}_{r, s_1 + \dots + s_n + n - 1}(r + s_1; Y_1, s_2; Y_2; \dots; s_n; Y_n)$$

and the coaction by “deconcatenation up to Y_1 .”

4.2.4. A left-action reformulation of the fusion. Another useful reformulation of fusion product (4.3) is

$$(s; Y) \odot (t; Z) = \tilde{\mathcal{L}}_{s+1, t}((s; Y) \otimes (t; Z)) \equiv (s; Y) \curlyvee (t; Z),$$

where we use the definition of $\tilde{\mathcal{L}}$ from **3.3.2**, and where *all the $s + 1$ strands in $(s; Y)$ “act” on $(t; Z)$* , i.e., $\tilde{\mathcal{L}}$ takes $X^{\otimes s} \otimes Y$ as its first argument.

4.3. Braiding of the fusion product of Yetter–Drinfeld modules.

4.3.1. Proposition. *For two $((Y|$ -type Yetter–Drinfeld modules, the braiding*

$$\mathcal{B} : ((Y| \odot ((Z| \rightarrow ((Z| \odot ((Y|$$

is given on braided components by the maps

$$\begin{aligned} \mathcal{B}_{s, t} : (s; Y; t; Z) &\rightarrow \bigoplus_{i=0}^s (i; Z; s - i + t; Y), \\ \mathcal{B}_{s, t} &= -\tilde{\mathcal{R}}_{s, t+1} \mathcal{S}_{t+1}^{\uparrow(s+1)}. \end{aligned}$$

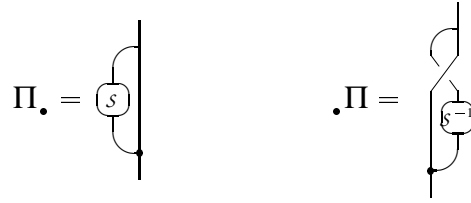
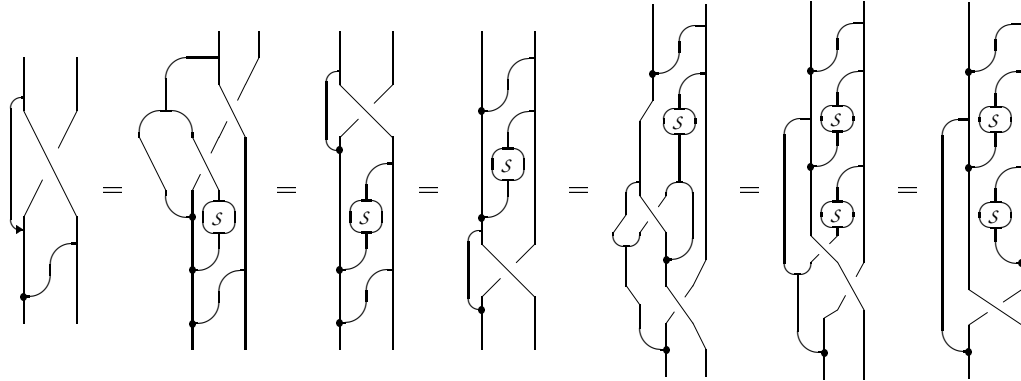


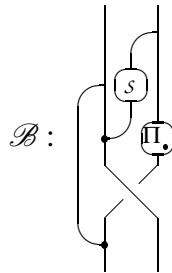
FIGURE 4.1. Two projectors on left coinvariants.

We emphasize that the antipode here formally acts, via (3.1), on all of the $t + 1$ strands of $(t; Z)$. All the $t + 1$ strands, moreover, then act from the right on $(s; Y)$ via an obvious extension of the right action $\tilde{\mathcal{H}} = \mathcal{V}'$ defined in 3.3.2.

4.3.2. Proof of 4.3.1. We find the braiding directly, by calculating how the standard Yetter–Drinfeld braiding (see (B.9)) is intertwined with the \mathcal{V} map (see B.3). For this, we represent $\mathcal{V} \circ (\text{standard Yetter–Drinfeld braiding})$ in a form $\mathcal{B} \circ \mathcal{V}$ via the following transformations:



(we used the action and coaction associativity and coassociativity to isolate and then kill a “bubble” as in (B.2) in the third diagram, and then insert such a bubble into the fourth). Everything below the \mathcal{V} -map at the top of the last diagram is therefore the desired braiding:



where $\Pi_.$ is defined in Fig. 4.1. But there is more to it, because $\Pi_.$ is a projector onto the

left coinvariants in each $((Y|:$

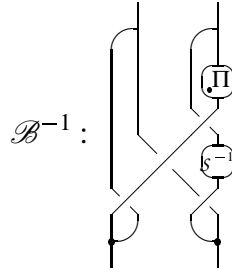
$$\Pi_{\bullet}(s; Y) = \begin{cases} 0, & s \geq 1, \\ (0; Y), & s = 0 \end{cases}$$

(by the nature of the coaction and of the $\tilde{\mathcal{L}}$ action, this property is just the one in (B.2)). It hence follows that on graded components, the braiding is given by

$$(4.6) \quad \mathcal{B}_{s,t} = \sum_{i=0}^s \Psi_{s-i+1,1}^{\uparrow i} \sqcup_{s-i,t}^{\uparrow i} \Psi_{1,t+1}^{\uparrow s} \mathcal{S}_t^{\uparrow(s+1)} : (s; Y; t; Z) \rightarrow \bigoplus_{i=0}^s (i; Z; s-i+t; Y).$$

Finally, because $\Psi_{1,t} \mathcal{S}_t^{\uparrow 1} = -\mathcal{S}_{t+1}$, it is not difficult to see that the braiding can be equivalently written as stated in the proposition. \square

4.3.3. The inverse braiding is found similarly, with the result

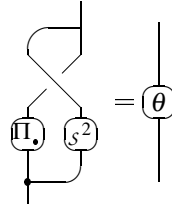


where Π_{\bullet} is also a projector on left coinvariants, defined in Fig. 4.1

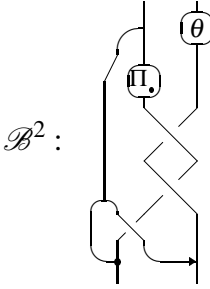
4.3.4. Squared braiding. Standard diagram manipulations allow calculating the squared braiding \mathcal{B}^2 . In terms of components, the \mathcal{B}^2 operation is by definition the map

$$\mathcal{B}^2 \equiv \sum_{i=0}^s \mathcal{B}_{i,s-i+t} \mathcal{B}_{i,s,t} : (s; Y; t; Z) \rightarrow \bigoplus_{i=0}^s (i; Y; s-i+t; Z),$$

where $\mathcal{B}_{i,s,t}$ is the summand in (4.6). The calculation standardly uses the axioms relating the “dotted” actions to left coaction and also the projector property of Π_{\bullet} , plus the identity



where θ is defined in (3.2) (and the identity itself is a reformulation of the one in 3.3.8). The result of diagram evaluation is

(4.7) $\mathcal{B}^2 :$ 

Componentwise, if the two input strands in the diagram represent $(s; Y)$ and $(t; Z)$, then the θ map here is exactly $\theta_{t+1}^{\uparrow(s+1)}$, the full twist of *all* the $t+1$ factors in $(t; Z) = X^{\otimes t} \otimes Z$. Furthermore, because Π_\bullet projects onto left coinvariants, we can reformulate the last diagram as the following proposition.

4.3.5. Proposition. \mathcal{B}^2 restricted to $(s; Y; t; Z)$ is the map

$$\sum_{i=0}^s \text{Ad}_{s-i,t}^{\uparrow(i+1)} \Psi_{s-i,1}^{\uparrow i} \theta_{1+t+1}^{\uparrow s} : (s; Y; t; Z) \rightarrow \bigoplus_{i=0}^s (i; Y; s-i+t; Z)$$

where $\theta_{1+t+1}^{\uparrow s} = \text{id}^{\otimes s} \otimes \theta_{t+2} : (s) \otimes (Y; t; Z) \rightarrow (s) \otimes (Y; t; Z)$ is the full twist acting on the last $t+2$ tensor factors.

The formula is worth being restated in words. To apply the squared braiding to $(s; Y; t; Z)$, we write this as $(s) \otimes (Y; t; Z)$ and act with the full twist on the “ribbon” $(Y; t; Z)$ whose “edges” are the two “vertex” linear spaces. Next, we deconcatenate (s) into $(i) \otimes (s-i)$ and then take $(s-i)$ past Y via braiding and evaluate the left adjoint action $(s-i) \blacktriangleright (t; Z)$.

4.4. Remark. In the general position assumed in Secs. (2) and (3), no two punctures w_1, \dots, w_m lie on the same line from the family. Degenerate cases where two punctures happen to be on the same line (“two fixed lines collide”) can be studied in terms of another homology complex, based on a stratification of the configuration space of the w_1, \dots, w_m , with the m punctures now “movable.” Similarly to what we did above, the differential of the homology complex with coefficients in the corresponding local system then yields an associative multiplication. The full-fledged geometric construction is not given here, but the fusion product can be regarded as its “algebraic counterpart.” On the algebraic side, in terms of the corresponding “large” Nichols algebra $\mathfrak{B}(Y)$, we note that the case of a subspace $X \subset Y$ such that $\Psi(X \otimes X) = X \otimes X$ was considered in [25], where, under some “nondegeneracy” assumptions, it was shown that a subalgebra K of $\mathfrak{B}(Y)$ exists such that $\mathfrak{B}(Y) = K \otimes \mathfrak{B}(X)$ as a left K -module and right $\mathfrak{B}(X)$ -module. Specifically, K is the kernel of braided derivations ∂_α for $\alpha \in X^* \subset Y^*$ (with the last inclusion defined in terms of dual bases), and $\mathfrak{B}(Y) = K \otimes \mathfrak{B}(X)$ as a left K -module and right $\mathfrak{B}(X)$ -module. This

description of $\mathfrak{B}(Y)$ as $K \otimes \mathfrak{B}(X)$ is very appealing from the CFT standpoint: the algebra of screenings has to be tensored just with pure (“unscreened”) vertex operators.

5. ONE-DIMENSIONAL X AND THE $(p, 1)$ LOGARITHMIC CFT MODELS

CFT applications are associated with *diagonal braiding*, which means that a basis (x_i) exists in the braided linear space X such that

$$(5.1) \quad \Psi : x_i \otimes x_j \mapsto q_{ij} x_j \otimes x_i$$

for some numbers q_{ij} . The braiding involving the Y space is also assumed diagonal (in addition to the q_{ij} , it is then defined by $q_{i\beta}$, $q_{\alpha j}$, and $q_{\alpha\beta}$, where α labels basis elements of Y). In Appendix C, we briefly describe the context in which diagonal braiding occurs in the theory of Nichols algebras. It has been in the focus of recent activity, with profound results (see [12, 23] and the references therein).

We specialize the constructions in Secs. 3–4 to the simple(st) case of a rank-1 Nichols algebra that in the CFT language corresponds to the $(p, 1)$ logarithmic conformal models [40, 41] (also see [42, 32, 45, 46] and the references therein). We thus take X to be the one-dimensional linear space spanned by a screening operator; the Y space is then spanned by highest-weight vectors of representations of the lattice vertex-operator algebra associated with the 1-dimensional lattice $\sqrt{2p}\mathbb{Z}$. We fix an integer $p \geq 2$ and set

$$(5.2) \quad q = e^{\frac{i\pi}{p}}.$$

5.1. CFT background.

5.1.1. Lattice vertex-operator algebra. Let $\phi(z)$ be a free chiral scalar field with the basic operator product expansion

$$\phi(z)\phi(w) = \log(z - w).$$

The space of observables \mathcal{F} is the direct sum $\mathcal{F} = \bigoplus_{j \in \mathbb{Z}} \mathcal{F}_j$ of Fock modules \mathcal{F}_j , $j \in \mathbb{Z}$, generated by $\partial\phi(z)$ from the vertices

$$(5.3) \quad V^j(z) \equiv V_{0,0}^j(z) = e^{\frac{j}{\sqrt{2p}}\phi(z)}, \quad j \in \mathbb{Z}$$

(the “double zero” notation is redundant at this point, but is introduced for uniformity in what follows). A vector from \mathcal{F}_j has the form $P(\partial\phi(z))V^j(z)$ for a differential polynomial P .

The maximum local algebra acting in \mathcal{F} is the lattice vertex-operator algebra $\mathcal{L}_{\sqrt{2p}}$ associated with the lattice $\sqrt{2p}\mathbb{Z}$. The space \mathcal{F} decomposes into a direct sum of $2p$ simple $\mathcal{L}_{\sqrt{2p}}$ -modules

$$\mathcal{F} = \bigoplus_{r \in \mathbb{Z}_{2p}} \mathcal{V}_r$$

where $\mathcal{V}_r = \bigoplus_{j \equiv r \pmod{2p}} \mathcal{F}_j$. The vacuum $\mathcal{L}_{\sqrt{2p}}$ -module is generated from $V^0 = 1$.

Exchange relations for the vertex operators from \mathcal{F} are

$$(5.4) \quad V^j(z)V^k(w) = q^{\frac{jk}{2}} V^k(w)V^j(z), \quad z > w,$$

where the condition $z > w$ is to be used for z and w lying on the same line from the family introduced in **2.1.1**, and is understood in the sense of the orientation chosen on the lines.

5.1.2. The $(p, 1)$ model and the triplet algebra. The $(p, 1)$ logarithmic CFT model is based on the triplet vertex operator algebra originating in [40, 41]. This algebra can be constructed by taking the kernel of the “short” screening operator in the vacuum representation of $\mathcal{L}_{\sqrt{2p}}$ [42]. The algebra is generated by three currents $W^-(z)$, $W^0(z)$, and $W^+(z)$ given by

$$(5.5) \quad W^-(z) = e^{-\sqrt{2p}\varphi(z)}, \quad W^0(z) = [S_+, W^-(z)], \quad W^+(z) = [S_+, W^0(z)],$$

where $S_+ = \int e^{\sqrt{2p}\varphi}$ (the “long” screening operator). The associated energy–momentum tensor

$$T(z) = \frac{1}{2} \partial \varphi(z) \partial \varphi(z) + \frac{p-1}{\sqrt{2p}} \partial^2 \varphi(z),$$

encodes the Virasoro algebra with central charge $c = 13 - 6p - \frac{6}{p}$.

The main source of Hopf-algebra structures associated with the model is the “short” screening operator

$$(5.6) \quad F(1) = \int_{-\infty < z < \infty} f(z), \quad f(z) = V_{0,0}^{-2}(z) = e^{-\sqrt{\frac{2}{p}}\varphi(z)}.$$

We let X be the 1-dimensional space with basis (5.6).

5.2. $\mathcal{H}(X)$ and $\mathfrak{B}(X)$. The braided Hopf algebra $\mathcal{H} = \mathcal{H}(X)$ is the span of the iterated screenings

$$(5.7) \quad F(r) = \int \cdots \int_{-\infty < z_1 < \cdots < z_r < \infty} s(z_1) \cdots s(z_r), \quad r \geq 1.$$

It follows from (5.4) that

$$(5.8) \quad \Psi(F(r) \otimes F(s)) = q^{2rs} F(s) \otimes F(r)$$

and

$$F(r) \cdot F(s) = \left\langle \begin{matrix} r+s \\ r \end{matrix} \right\rangle F(r+s),$$

where the q -binomial coefficients defined as

$$(5.9) \quad \left\langle \begin{matrix} r \\ s \end{matrix} \right\rangle = \frac{\langle r \rangle!}{\langle s \rangle! \langle r-s \rangle!}, \quad \langle r \rangle! = \langle 1 \rangle \cdots \langle r \rangle, \quad \langle r \rangle = \frac{q^{2r} - 1}{q^2 - 1}$$

are specialized to $q = q$. In particular, $\langle p \rangle = 0$. The deconcatenation coproduct is

$$\Delta(F(r)) = \sum_{s=0}^r F(s) \otimes F(r-s),$$

and the antipode follows by evaluating (3.1) for the above braiding:

$$\mathcal{S}(F(r)) = (-1)^r q^{r(r-1)} F(r).$$

The counit is merely $\varepsilon(F(r)) = \delta_{r,0}1$.

The Nichols algebra of (X, Ψ) — the subalgebra $\mathfrak{B}(X)$ in $\mathcal{H}(X)$ generated by $F(1)$ — is therefore the linear span of the $F(r)$ with $0 \leq r \leq p-1$.

We also write $F = F(1)$; then $F^p = 0$ and the algebra \mathfrak{B} has the basis F^i , $0 \leq i \leq p-1$.

5.2.1. The Y space. We next specify the Y space, the source of Yetter–Drinfeld modules over the Nichols algebra.

Setting $V^j = V^j(0)$, we define the braided linear space

$$Y = \text{span}(V^j \mid j \in \mathbb{Z}_{4p})$$

where the braiding is

$$(5.10) \quad \Psi(V^j \otimes V^k) = q^{\frac{jk}{2}} V^k \otimes V^j.$$

Also, the braiding of (elements of) X and Y is

$$(5.11) \quad \Psi(V^j \otimes F(r)) = q^{-jr} F(r) \otimes V^j, \quad \Psi(F(r) \otimes V^j) = q^{-jr} V^j \otimes F(r).$$

5.3. Hopf bimodules and Yetter–Drinfeld $\mathfrak{B}(X)$ -modules.

5.3.1. Hopf bimodules. For diagonal braiding, the Hopf bimodule $((Y))$ (see 3.2) decomposes as $((Y)) = \bigoplus_i ((y_i))$, where the sum is taken over a basis in Y that diagonalizes the braiding. Hence, a Hopf $\mathfrak{B}(X)$ -bimodule $((V^j))$ is generated from each of the $4p$ chosen basis elements of Y . For each $j = 1, \dots, 4p$, the basis in $((V^j))$ is

$$(5.12) \quad V_{r,t}^j \equiv (F(r); V^j; F(t)) = \int \cdots \int_{-\infty < x_1 < \cdots < x_r < 0 < x_{r+1} < \cdots < x_{r+t} < \infty} s(x_1) \cdots s(x_r) V^j(0) s(x_{r+1}) \cdots s(x_{r+t}), \quad 0 \leq r, t \leq p-1.$$

In this basis, the left and right $\mathfrak{B}(X)$ -actions (see (3.3) and (3.4)) become

$$\begin{aligned} F \cdot V_{r,t}^j(z) &= \langle r+1 \rangle V_{r+1,t}^j(z) + q^{2r-j} \langle t+1 \rangle V_{r,t+1}^j(z), \\ V_{r,t}^j(z) \cdot F &= q^{2t-j} \langle r+1 \rangle V_{r+1,t}^j(z) + \langle t+1 \rangle V_{r,t+1}^j(z). \end{aligned}$$

The left and right coactions on $((V^j))$ are given, as usual, by deconcatenation of the integrals in (5.12):

$$(5.13) \quad \delta_L V_{s,t}^j = \sum_{j=0}^s F(j) \otimes V_{s-j,t}^j, \quad \delta_R V_{s,t}^j = \sum_{j=0}^t V_{s,t-j}^j \otimes F(j).$$

5.3.2. “Multivertex” Hopf bimodules. The multivertex Hopf bimodules introduced in 3.4 are now labeled by $\mathbf{j} = \{j_1, j_2, \dots, j_\ell\}$, where each j_i ranges over $4p$ values. For each such \mathbf{j} , the Hopf bimodule $((\mathbf{j}))$ is the span of the $V_{t_1, t_2, \dots, t_{\ell+1}}^{\mathbf{j}}$ that are obtained by fixing, once and for all, some $z_1 < z_2 < \dots < z_\ell$ in the expressions

$$V_{t_1, t_2, \dots, t_{\ell+1}}^{\mathbf{j}}(z_1, z_2, \dots, z_\ell) = (F(t_1); V^{j_1}(z_1); F(t_2); V^{j_2}(z_2); \dots; F(t_\ell); V^{j_\ell}(z_\ell); F(t_{\ell+1})),$$

where $0 \leq t_j \leq p-1$. These expressions are $(t_1 + t_2 + \dots + t_{\ell+1})$ -fold integrals:

$$= \int \cdots \int \prod_{n=1}^{\ell} \left(\prod_{m=1}^{t_n} (s(x_m^n)) V^{j_n}(z_n) \right) \prod_{m=1}^{t_{\ell+1}} (s(x_m^{\ell+1}))$$

$$\begin{aligned} & -\infty < x_1^1 < \dots < x_{t_1}^1 < z_1 < x_1^2 < \dots \\ & \dots < x_{t_\ell}^\ell < z_\ell < x_1^{\ell+1} < \dots < x_{t_{\ell+1}}^{\ell+1} < \infty \end{aligned}$$

with the integrations only over the $x_{t_i}^j$, *not over the z_i* .

The structure of the $\mathfrak{B}(X)$ -action on $V_{t_1, t_2, \dots, t_{\ell+1}}^{\mathbf{j}} \in ((\mathbf{j}))$ is already clearly seen in the case of two-vertex modules ($\ell = 2$). The left and right $\mathfrak{B}(X)$ -actions are

$$\begin{aligned} F \cdot V_{n_1, n_2, n_3}^{\{j_1, j_2\}} &= \langle n_1 + 1 \rangle V_{n_1+1, n_2, n_3}^{\{j_1, j_2\}} + q^{2n_1-j_1} \langle n_2 + 1 \rangle V_{n_1, n_2+1, n_3}^{\{j_1, j_2\}} \\ &\quad + q^{2n_1+2n_2-j_1-j_2} \langle n_3 + 1 \rangle V_{n_1, n_2, n_3+1}^{\{j_1, j_2\}}, \\ V_{n_1, n_2, n_3}^{\{j_1, j_2\}} \cdot F &= q^{2n_2+2n_3-j_1-j_2} \langle n_1 + 1 \rangle V_{n_1+1, n_2, n_3}^{\{j_1, j_2\}} + q^{2n_3-j_2} \langle n_2 + 1 \rangle V_{n_1, n_2+1, n_3}^{\{j_1, j_2\}} \\ &\quad + \langle n_3 + 1 \rangle V_{n_1, n_2, n_3+1}^{\{j_1, j_2\}}, \end{aligned}$$

5.3.3. Yetter–Drinfeld $\mathfrak{B}(X)$ -modules. As in 3.3, we take the space of right coinvariants in $((V^j))$, that is,

$$\mathcal{V}^j = \text{span}(V_{s,0}^j \mid 0 \leq s \leq p-1).$$

It is a (left–left) Yetter–Drinfeld $\mathfrak{B}(X)$ -module under the left coaction (5.13) and the left adjoint action described in 3.3.3, which now evaluates as

$$(5.14) \quad F(r) \blacktriangleright V_{s,0}^j = \left\langle \begin{matrix} r+s \\ r \end{matrix} \right\rangle \prod_{a=s}^{s+r-1} (1 - q^{2a-2j}) V_{r+s,0}^j.$$

5.4. The module nomenclature. The dependence of the right-hand side of (5.4) (and hence (5.10)) on j and k is through $q^{\frac{jk}{2}}$, and is therefore $4p$ -periodic. By contrast, it is clear from (5.14) that the module structure on \mathcal{V}^j depends on j modulo p (and in fact so does the comodule structure).

Hence, if the braiding structure is forgotten, then there are only p nonisomorphic $\mathfrak{B}(X)$ module comodules among the \mathcal{V}^j (e.g., those with $j = 0, \dots, p-1$). If not only the module comodule structure but also the braiding $\Psi : \mathfrak{B}(X) \otimes \mathcal{V}^j \rightarrow \mathcal{V}^j \otimes \mathfrak{B}(X)$ is considered, then there are $2p$ nonisomorphic objects among the \mathcal{V}^j . And finally, if these are considered as objects in the braided category of Yetter–Drinfeld $\mathfrak{B}(X)$ -modules, then $4p$ of them are nonisomorphic.

It is therefore convenient to parameterize the “momenta” j as

$$(5.15) \quad j \equiv r - 1 - vp \pmod{4p}, \quad r = 1, \dots, p, \quad v \in \mathbb{Z}_4 = \{0, 1, 2, 3\}$$

to single out the range $r = 1, \dots, p$ that is only “seen” by the representation theory with no braiding considered, and set

$$\mathcal{V}_r(v) = \mathcal{V}^{r-1-vp}, \quad r = 1, \dots, p, \quad v \in \mathbb{Z}_4.$$

The following statements are verified directly, using (5.14) and (5.13).

- (1) *The Yetter–Drinfeld modules $\mathcal{V}_p(v)$, $v \in \mathbb{Z}_4$, are simple.*
- (2) *For $1 \leq r \leq p-1$, the Yetter–Drinfeld module $\mathcal{V}_r(v)$ contains a simple r -dimensional Yetter–Drinfeld submodule*

$$\mathcal{X}_r(v) = \text{span}(V_{s,0}^{r-1-vp} \mid 0 \leq s \leq r-1).$$

- (3) *The quotient $\mathcal{V}_r(v)/\mathcal{X}_r(v)$ is isomorphic to $\mathcal{X}_{p-r}(v+1)$ as an object in the braided category of Yetter–Drinfeld $\mathfrak{B}(X)$ -modules.*

5.5. Fusion product and related structures. Formula (4.3) for the fusion product of Yetter–Drinfeld modules takes the form

$$V_{r,0}^j \odot V_{s,0}^m = \sum_{i=0}^s q^{-ij} \begin{Bmatrix} r+i \\ r \end{Bmatrix} V_{r+i, s-i, 0}^{\{j, m\}}.$$

The $V_{r_1+i, s_1-i, 0}^{\{j, m\}}$ occurring in the right-hand side are elements of the two-vertex Yetter–Drinfeld modules (see 4.2.3). The left adjoint action on these is most easily calculated from the factorization formula (4.5): it follows that

$$(5.16) \quad F(r) \blacktriangleright V_{t_1, t_2, 0}^{\{j_1, j_2\}} = \sum_{s=0}^r q^{s(2t_1-j_1)} \begin{Bmatrix} t_1+r-s \\ r-s \end{Bmatrix} \begin{Bmatrix} t_2+s \\ s \end{Bmatrix} \\ \times \prod_{a=s+1}^r (1 - q^{2(t_1+2t_2+a-1-j_1-j_2)}) \prod_{b=0}^{s-1} (1 - q^{2(t_2+b-j_2)}) V_{t_1+r-s, t_2+s, 0}^{\{j_1, j_2\}}.$$

Calculations with these ingredients support the following conjecture.

5.5.1. Conjecture. The fusion products of the $\mathcal{X}_r(\nu)$, $1 \leq r \leq p$, $\nu \in \mathbb{Z}_4$, decompose as

$$(5.17) \quad \mathcal{X}_r(\nu) \odot \mathcal{X}_s(\mu) = \bigoplus_{j=|r-s|+1}^{\min(r+s-1, 2p-r-s-1)} \mathcal{X}_j(\nu + \mu) \oplus \bigoplus_{j=2p-r-s+1}^p \mathcal{P}_j(\nu + \mu),$$

where both summations are with step 2 (and start at the lower limit, which may be essential for the second sum), and the $\mathcal{P}_j(\nu)$ modules are as follows: $\mathcal{P}_p(\nu) = \mathcal{X}_p(\nu)$, and each $\mathcal{P}_s(\nu)$ with $s = 1, 2, \dots, p-1$ is an indecomposable Yetter–Drinfeld $\mathfrak{B}(X)$ -module with the structure of subquotients given by

$$(5.18) \quad \begin{array}{ccc} & \mathcal{X}_s(\nu) & \\ \swarrow & & \searrow \\ \mathcal{X}_{p-s}(\nu-1) & & \mathcal{X}_{p-s}(\nu+1) \\ \searrow & & \swarrow \\ & \mathcal{X}_s(\nu) & \end{array}$$

We illustrate this with examples.

5.5.2. Examples. We restrict ourself to $p = 3$ and consider $\mathcal{X}_3(0) \odot \mathcal{X}_3(0)$ and $\mathcal{X}_2(0) \odot \mathcal{X}_3(0)$.

5.5.2.1. $\mathcal{X}_3(0) \odot \mathcal{X}_3(0)$. For $p = 3$, we choose $V_{r,s,0}^{\{2,2\}}$ with $0 \leq r, s \leq 2$ as a basis in $\mathcal{X}_3(0) \odot \mathcal{X}_3(0)$. Using (5.16) and (5.13), we then verify that the subspace $\mathcal{P}_1(0)$ spanned by the vectors

$$(5.19) \quad V_{0,0,0}^{\{2,2\}}, \quad V_{1,0,0}^{\{2,2\}} - V_{0,1,0}^{\{2,2\}}, \quad V_{2,0,0}^{\{2,2\}} - V_{1,1,0}^{\{2,2\}}, \quad V_{1,2,0}^{\{2,2\}}, \quad V_{2,2,0}^{\{2,2\}}, \quad V_{0,2,0}^{\{2,2\}},$$

is invariant under action (5.16) and the left coaction in (5.13), and is therefore a Yetter–Drinfeld submodule in $\mathcal{X}_3(0) \odot \mathcal{X}_3(0)$. Moreover, the vectors are arranged in accordance with the structure claimed in (5.18). For example, coaction applied to the top vector yields $1 \otimes (V_{2,0,0}^{\{2,2\}} - V_{1,1,0}^{\{2,2\}}) + F(1) \otimes (V_{1,0,0}^{\{2,2\}} - V_{0,1,0}^{\{2,2\}}) + F(2) \otimes V_{0,0,0}^{\{2,2\}}$, which is an element of $\mathfrak{B}(X) \otimes \mathcal{P}_1(0)$.⁸

Another Yetter–Drinfeld submodule in $\mathcal{X}_3(0) \odot \mathcal{X}_3(0)$ is the span of

$$(5.20) \quad V_{0,1,0}^{\{2,2\}}, \quad V_{1,1,0}^{\{2,2\}} + V_{0,2,0}^{\{2,2\}}, \quad V_{2,1,0}^{\{2,2\}} + V_{1,2,0}^{\{2,2\}}$$

Using (5.16) and (5.13), it is straightforward to verify that this last subspace is in fact isomorphic to the Yetter–Drinfeld module $\mathcal{X}_3(0)$. Hence,

$$\mathcal{X}_3(0) \odot \mathcal{X}_3(0) = \mathcal{P}_1(0) \oplus \mathcal{X}_3(0),$$

⁸To check that the parameters ν of subquotients in (5.19) are as shown in (5.18), we represent the total momenta of the relevant vertices in form (5.15). Because the elements of the module are related by either the action or the coaction, it is straightforward to see that the ν -pattern is always as shown in (5.18). It follows similarly that the ν parameter is always additive in (5.17).

which is (5.17) in this particular case.

5.5.2.2. $\mathcal{X}_2(0) \odot \mathcal{X}_3(0)$. This space has the basis $V_{r,s,0}^{\{2,2\}}$, with $0 \leq r \leq 1$ and $0 \leq s \leq 2$. We suggestively arrange these vectors as

$$(5.21) \quad \begin{array}{ccccc} V_{0,0,0}^{\{1,2\}}, & V_{1,0,0}^{\{1,2\}}, & \langle 2 \rangle V_{2,0,0}^{\{1,2\}} + qV_{1,1,0}^{\{1,2\}}, & & qV_{2,1,0}^{\{1,2\}} + \langle 2 \rangle V_{1,2,0}^{\{1,2\}} \\ & V_{0,1,0}^{\{1,2\}}, & qV_{1,1,0}^{\{1,2\}} + \langle 2 \rangle V_{0,2,0}^{\{1,2\}}, & & \end{array}$$

and then use (5.16) and (5.13) to verify that this is an indecomposable Yetter–Drinfeld module, $\mathcal{P}_2(0)$ (again, with the structure following the pattern in (5.18)), and hence

$$\mathcal{X}_2(0) \odot \mathcal{X}_3(0) = \mathcal{P}_2(0),$$

again in accordance with (5.17).

5.6. \odot is not commutative. We next show that if the $\mathfrak{B}(X)$ modules constructed above are considered not as objects in the braided category of Yetter–Drinfeld modules but as only $\mathfrak{B}(X)$ module comodules with the braiding $\Psi : \mathfrak{B}(X) \otimes \mathcal{V}^j \rightarrow \mathcal{V}^j \otimes \mathfrak{B}(X)$ (and with the corresponding morphisms; see 5.4), then the \odot product is not commutative, i.e., the product of two modules taken in two different orders may result in nonisomorphic modules. As in [69], showing this requires modules of the “ \mathcal{O} ” type, labeled by a parameter $z \in \mathbb{CP}^1$. Such a $\mathfrak{B}(X)$ module comodule, denoted by $\mathcal{O}_2(0)(1, z)$, is readily identified as a sub(co)module in (5.19): for each $z = z_1 : z_2$, the relevant subspace is the span of

$$\begin{aligned} t_1 &= z_1 V_{0,0,0}^{\{2,2\}} + z_2 V_{1,2,0}^{\{2,2\}}, & t_2 &= z_1 (V_{1,0,0}^{\{2,2\}} - V_{0,1,0}^{\{2,2\}}) + z_2 V_{2,2,0}^{\{2,2\}}, \\ b &= V_{0,2,0}^{\{2,2\}}. \end{aligned}$$

That this is a submodule subcomodule follows from the action and coaction formulas

$$\begin{aligned} F \blacktriangleright t_1 &= (1 - q^{-2})t_2, & F \blacktriangleright t_2 &= -z_1(1 - q^2)b, & F \blacktriangleright b &= 0, \\ \delta_L t_1 &= 1 \otimes t_1 + z_2 F(1) \otimes b, & \delta_L t_2 &= 1 \otimes t_2 + F(1) \otimes t_1 + z_2 F(2) \otimes b, & \delta_L b &= 1 \otimes b. \end{aligned}$$

We now calculate $\mathcal{X}_1(1) \odot \mathcal{O}_2(0)(1, z)$ and $\mathcal{O}_2(0)(1, z) \odot \mathcal{X}_1(1)$. From the action of F on three-vertex objects

$$(5.22) \quad \begin{aligned} F \blacktriangleright V_{t_1, t_2, t_3, 0}^{\{a, b, c\}} &= (1 - q^{2t_1 + 4t_2 + 4t_3 - 2a - 2b - 2c}) \langle t_1 + 1 \rangle V_{t_1 + 1, t_2, t_3, 0}^{\{a, b, c\}} \\ &\quad + q^{2t_1 - a} (1 - q^{2t_2 + 4t_3 - 2b - 2c}) \langle t_2 + 1 \rangle V_{t_1, t_2 + 1, t_3, 0}^{\{a, b, c\}} \\ &\quad + q^{2t_1 + 2t_2 - a - b} (1 - q^{2t_3 - 2c}) \langle t_3 + 1 \rangle V_{t_1, t_2, t_3 + 1, 0}^{\{a, b, c\}}, \end{aligned}$$

it is straightforward to see that

$$(5.23) \quad \mathcal{X}_1(1) \odot \mathcal{O}_2(0)(1, z) = \mathcal{O}_2(1)(1, -z), \quad \mathcal{O}_2(0)(1, z) \odot \mathcal{X}_1(1) = \mathcal{O}_2(1)(1, z).$$

The \odot product is therefore not commutative. Another way to see this is to consider the braiding \mathcal{B} from 4.3.1. On two-vertex Yetter–Drinfeld modules, the braiding operation \mathcal{B} from 4.3.1 becomes

$$\mathcal{B}V_{s,t,0}^{\{j_1,j_2\}} = (-1)^t q^{\frac{1}{2}j_1j_2 - t(j_1+j_2) + t(t-1)} \sum_{r=0}^s q^{-j_2r} \left\langle \begin{matrix} t+r \\ t \end{matrix} \right\rangle V_{s-r,t+r,0}^{\{j_2,j_1\}}.$$

A simple calculation with 4.3.1 and (5.22) shows that

$$(5.24) \quad \mathcal{B} : \mathcal{X}_1(1) \odot \mathcal{O}_2(0)(1, z) \rightarrow \mathcal{O}_2(0)(1, -z) \odot \mathcal{X}_1(1)$$

is an isomorphism of module comodules.

The braided Hopf algebra $\mathfrak{B}(X)$ admits an outer automorphism defined as

$$\alpha : F(r) \mapsto (-1)^r F(r)$$

which acts on basis elements of $\mathfrak{B}(X)$ -modules as

$$\alpha : V_{s_1,s_2,\dots,s_\ell,0}^{\mathbf{j}} \mapsto (-1)^{\sum_{m=1}^{\ell} s_m} V_{s_1,s_2,\dots,s_\ell,0}^{\mathbf{j}},$$

where $\mathbf{j} = \{j_1, j_2, \dots, j_\ell\}$ and each j_i ranges over \mathbb{Z}_{4p} . It is immediate to see that this preserves the Yetter–Drinfeld condition (B.7), and also that

$$\alpha : \mathcal{X}_r(v) \rightarrow \mathcal{X}_r(v), \quad \alpha : \mathcal{P}_r(v) \rightarrow \mathcal{P}_r(v),$$

but

$$\alpha : \mathcal{O}_2(0)(1, z) \rightarrow \mathcal{O}_2(0)(1, -z).$$

We can then write (5.24) as

$$(5.25) \quad \mathcal{B} : \mathcal{X}_1(1) \odot \mathcal{O}_2(0)(1, z) \rightarrow \alpha(\mathcal{O}_2(0)(1, z)) \odot \alpha(\mathcal{X}_1(1)).$$

The representation category of the triplet W -algebra in $(p, 1)$ models is equivalent to the representation category of the $\overline{\mathcal{U}}_q s\ell(2)$ quantum group [43, 46] as Abelian categories. But some indecomposable $\overline{\mathcal{U}}_q s\ell(2)$ representations (the “ \mathcal{O} ” modules in [43], from which the notation was borrowed above) demonstrate “noncommutativity” (tensor product in two different orders gives nonisomorphic modules) [69]. Not surprisingly, this is “the same” noncommutativity as in (5.23). The category of Yetter–Drinfeld $\mathfrak{B}(X)$ -modules with the braiding among them forgotten (and with the morphisms being module comodule morphisms commuting with the braiding $\Psi : \mathfrak{B}(X) \otimes \mathcal{V}^j \rightarrow \mathcal{V}^j \otimes \mathfrak{B}(X)$) is equivalent to the category of representations of the triplet W -algebra as an Abelian category.

In another form, the relation to $(p, 1)$ models can be expressed in terms of “categories with monodromy” (*entwined* categories in [70])—which on the algebraic side is the category of Yetter–Drinfeld $\mathfrak{B}(X)$ -modules \mathcal{V} endowed not with braiding but with the squared braiding (monodromy) $\Psi^2 : \mathcal{V} \otimes \mathcal{W} \rightarrow \mathcal{V} \otimes \mathcal{W}$ (and with the braiding $\Psi : \mathfrak{B}(X) \otimes \mathcal{V} \rightarrow \mathcal{V} \otimes \mathfrak{B}(X)$).

5.7. The dual triangular part of the algebra. We recall the “braided double problem” mentioned in the Introduction (see p. 8)—the nonexistence of an immediate analogue of the Drinfeld double in the braided case. We have also noted that constructing an ordinary Hopf algebra via double bosonization requires a list of conditions (cf. [61]). For our current $\mathfrak{B}(X)$ algebra of a very simple structure, we now show that introducing the dual generator as in 3.5 at least leads to an associative algebra.

Considering \mathfrak{B}^* , we introduce $E \in \mathfrak{B}^*$ as the linear functional

$$(5.26) \quad \langle E, F(r) \rangle = \delta_{r,1}.$$

A basis in \mathfrak{B}^* is then E^n with $0 \leq n \leq p-1$.

The action of E on Hopf bimodule elements (5.12), also denoted by \blacktriangleright for uniformity and defined by $E \blacktriangleright V = \langle E, V_{(-1)} \rangle V_{(0)}$ with coaction (5.13), is

$$(5.27) \quad E \blacktriangleright V_{s_1, s_2}^j = V_{s_1-1, s_2}^j.$$

Then the “commutator” evaluated in 3.5.1 becomes

$$(5.28) \quad ((E \blacktriangleright)(F \blacktriangleright) - q^2(F \blacktriangleright)(E \blacktriangleright))V_{s_1, s_2}^j = (1 - q^{4s_1+4s_2-2j})V_{s_1, s_2}^j.$$

Therefore, \mathcal{K}_2 defined in 3.5.1 can be assigned the following action on elements of the corresponding Hopf bimodules (or, for $s_2 = 0$, Yetter–Drinfeld modules):

$$(5.29) \quad \mathcal{K}_2 \blacktriangleright V_{s_1, s_2}^j = q^{4s_1+4s_2-2j}V_{s_1, s_2}^j.$$

We then have the q -commutator relation (omitting the \blacktriangleright)

$$(5.30) \quad EF - q^2FE = 1 - \mathcal{K}_2.$$

Remarkably, it also follows from (5.29) that

$$(5.31) \quad \mathcal{K}_2 F = q^4 F \mathcal{K}_2, \quad \mathcal{K}_2 E = q^{-4} E \mathcal{K}_2.$$

We thus have an *associative algebra* $D(\mathfrak{B})$ generated by F , \mathcal{K}_2 , and E . (In addition, $F^p = 0$ as we saw above, $E^p = 0$ either from (5.27) or from the fact that $\mathfrak{B}(X^*) = \mathfrak{B}(X)^*$, and $\mathcal{K}_2^p = 1$ from (5.29)).

Interestingly, the operator \mathcal{K}_2 read off from (5.28) turns out to be “too coarse-grained” compared with what we have in the nonbraided Hopf algebra that is the Kazhdan–Lusztig-dual to the triplet W algebra—the $\overline{\mathcal{U}}_q \mathfrak{sl}(2)$ quantum group [32]. In fact, extracting a “root” K of \mathcal{K}_2 , $K^2 = \mathcal{K}_2$, allows making $D(B)$ into the $\overline{\mathcal{U}}_q \mathfrak{sl}(2)$ quantum group; for this, we just take KE and F as generators.

The relevant setting under which the occurrence of a nonbraided quantum group, such as $\overline{\mathcal{U}}_q \mathfrak{sl}(2)$, from a braided Hopf algebra, such as our $\mathfrak{B}(X)$, would not seem accidental is offered by “double bosonization” in the version described in [61] (also see [71]), which generalizes the Radford biproduct [62] (“bosonization” [63]) to triple “smash” products $A \# H \# B$, where A and B are Hopf algebras in ${}^H_H \mathcal{YD}$ dual to each other and the Hopf

algebra H is commutative and cocommutative. The relevant (and rather intricate) details are discussed in [61]; a crucial necessary requirement is that the braiding (C.2) in ${}^H_H\mathcal{YD}$ coincide with the braiding $u \otimes v \mapsto v_{(0)} \otimes v_{(-1)} \triangleright u$, which is the braiding in \mathcal{YD}_H^H for the right–right Yetter–Drinfeld structure $v \triangleleft h := s^{-1}(h) \triangleright v$ and $v_{(0)} \otimes v_{(1)} := v_{(0)} \otimes s(v_{(-1)})$. That is, the condition is $u_{(-1)} \triangleright v \otimes u_{(0)} = v_{(0)} \otimes v_{(-1)} \triangleright u$, which for diagonal braiding means that $q_{ij} = q_{ji}$.

6. CONCLUSIONS

We have outlined a strategy to construct Hopf algebra counterparts of vertex-operator algebras in logarithmic CFT. Part of the original motivation was to eliminate, or “resolve,” the ad hoc elements of the approach in [32, 43, 33, 34]. The “purest and minimalist” scheme appears to be the one based on *braided* Hopf algebras; relation to ordinary Hopf algebras should then follow via some form of bosonization. The idea is therefore that the “braided” approach is to play a more fundamental role in the duality of logarithmic CFT models with Hopf algebras/quantum groups.

In addition to its “strategic” significance, the braided ideology offers several attractive possibilities at a more practical level. For example, the square of the braiding \mathcal{B} calculated in Sec. 4.3.5 gives a powerful tool for constructing the center of the category: in a rigid category, one of the two lines in (4.7) can be “closed” using evaluation and coevaluation, with the result representing a central element action on the module corresponding to the other line. We expect to return to this in the future.

As we have noted, the choice of the basic contour system in Fig. 1.1 can be interpreted as a space–time decomposition of the complex plane, with the lines interpreted as spatial slices and a transverse direction as time. Topologically equivalent systems of contours give isomorphic Nichols algebras of screenings. It might be interesting to consider different nonequivalent choices of contour systems, among which the one that immediately suggests itself is the system of concentric circles, corresponding to radial quantization in CFT.

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APPENDIX A. BRAIDS AND SHUFFLES

We here recall the definitions pertaining to the braid group and “quantum” shuffles, and fix our conventions. In the last subsection in this appendix, A.6, we write a number of quantum shuffle identities that express the properties of Hopf and Yetter–Drinfeld modules established in Secs. 3 and 4.

A.1. “Classical” shuffles. Let π be a partition of $n \in \mathbb{N}$. We write $\pi = (\pi_1, \dots, \pi_k)$, where $\pi_1 \cup \dots \cup \pi_k = [1, \dots, n]$. The subgroup $\mathbb{S}_\pi^n \subset \mathbb{S}_n$ of *shuffle permutations* subordinated to π consists of those functions $\mathbb{S}_n \ni \sigma : [1, \dots, n] \rightarrow [1, \dots, n]$ that are monotonically increasing on each π_i .

A.2. Braids and “quantum” shuffles. Let $(\mathcal{C}, \otimes, \Psi)$ be a braided monoidal category. This means, in particular, that Ψ satisfies the braid equation (the hexagon equation in disguise, with the associativity morphisms omitted)

$$(\Psi \otimes \text{id})(\text{id} \otimes \Psi)(\Psi \otimes \text{id}) = (\text{id} \otimes \Psi)(\Psi \otimes \text{id})(\text{id} \otimes \Psi).$$

We recall that the Matsumoto section $\mathbb{S}_n \rightarrow \mathbb{B}_n$ is constructed as follows. Each $\sigma \in \mathbb{S}_n$ decomposes into a product of elementary permutations: $\sigma = \prod \tau_{i,i+1}$. Whenever this product is *of the minimum length*, $\sigma \mapsto \prod \Psi_{i,i+1}$.

For n objects X_i in $(\mathcal{C}, \otimes, \Psi)$ and a partition π of n , the (“quantum”) shuffle $\mathbb{W}_\pi^n \in \text{End}(X_1 \otimes \dots \otimes X_n)$ is

$$(A.1) \quad \mathbb{W}_\pi^n = \mathbb{W}_\pi^n[\mathcal{C}] = \sum_{\sigma \in \mathbb{S}_\pi^n} \sigma_{\mathcal{C}},$$

where $\sigma \mapsto \sigma_{\mathcal{C}} : \mathbb{S}_n \rightarrow \mathbb{B}_n \rightarrow \text{End}(X_1 \otimes \dots \otimes X_n)$ is the composition of the Matsumoto section and the braid group representation on $X_1 \otimes \dots \otimes X_n$, and $\ell(\sigma)$ is the reduced length of a $\sigma \in \mathbb{S}_n$. We omit \mathcal{C} from the notation for the \mathbb{W} . In fact, we mostly think of the shuffles in (A.1) and similar formulas below in “universal” terms, as elements of the braid group algebra (for braids on a “sufficiently large” number of strands, i.e., for an inductive-limit braid group $\mathbb{B}_\infty = \varinjlim \mathbb{B}_n$).

A.3. Notation. In a braided monoidal category $(\mathcal{C}, \otimes, \Psi)$, we use the standard notation

$$\begin{aligned} \Psi_i &\equiv \text{id}^{\otimes(i-1)} \otimes \Psi \otimes \text{id}^{\otimes(n-i-1)} : X_1 \otimes \dots \otimes X_n \\ &\rightarrow X_1 \otimes \dots \otimes X_{i-1} \otimes X_{i+1} \otimes X_i \otimes X_{i+2} \otimes \dots \otimes X_n, \end{aligned}$$

The braid equation is then expressed as

$$\Psi_s \Psi_{s+1} \Psi_s = \Psi_{s+1} \Psi_s \Psi_{s+1}.$$

The braiding of $X_1 \otimes \dots \otimes X_m$ with $Y_1 \otimes \dots \otimes Y_n$ is effected by

$$\Psi_{m,n} = (\Psi_n \dots \Psi_1)(\Psi_{n+1} \dots \Psi_2) \dots (\Psi_{n+m-1} \dots \Psi_m).$$

Another useful notation is the shift $\hat{\uparrow}^m$: for any morphism of the form $f = \text{id}^{\otimes i} \otimes F \otimes \text{id}^{\otimes j}$, we set

$$f^{\hat{\uparrow}^m} = \text{id}^{\otimes(i+m)} \otimes F \otimes \text{id}^{\otimes(j-m)}.$$

The two notational conventions are related via $\Psi_i^{\hat{\uparrow}^j} = \Psi_{i+j}$.

A.4. Braided factorial and binomials. Particular cases of quantum shuffles are the “braided binomials” and “braided factorials” (total symmetrizers)

$$\mathbb{W}_{r,s} = \mathbb{W}_{(r,s)}^{r+s} \in \text{End}(X^{\otimes(r+s)}) \quad \text{and} \quad \mathfrak{S}_n = \mathbb{W}_{(1,\dots,1)}^n \in \text{End}(X^{\otimes n}),$$

associated with the respective partitions $(r+s) = r+s$ and $n = 1 + \dots + 1$. The braided binomials satisfy (and are in fact determined by) the recursive relations

$$(A.2) \quad \mathbb{W}_{r,s+1} = \mathbb{W}_{r,s} + \mathbb{W}_{r-1,s+1} \Psi_{1,s+1}^{\hat{\uparrow}^{(r-1)}}$$

$$(A.3) \quad \mathbb{W}_{r+1,s} = \mathbb{W}_{r,s}^{\hat{\uparrow}^1} + \mathbb{W}_{r+1,s-1}^{\hat{\uparrow}^1} \Psi_{r+1,1}$$

(with $\mathbb{W}_{0,r} = \mathbb{W}_{r,0} = \text{id}$). Among the great many relations satisfied by these maps, we note

$$(A.4) \quad \mathbb{W}_{m+n,k} \mathbb{W}_{m,n} = \mathbb{W}_{m,n+k} \mathbb{W}_{n,k}^{\hat{\uparrow}^m},$$

$$(A.5) \quad \mathfrak{S}_{m+n} = \mathbb{W}_{m,n} \mathfrak{S}_m \mathfrak{S}_n^{\hat{\uparrow}^m},$$

and

$$\sum_{i=0}^r \mathbb{W}_{i,s} \mathbb{W}_{r-i,t-1}^{\hat{\uparrow}^{(i+s+1)}} \Psi_{r-i,s+1}^{\hat{\uparrow}^i} = \mathbb{W}_{r,s+t},$$

$$\sum_{j=0}^t \mathbb{W}_{s,j} \Psi_{1,j}^{\hat{\uparrow}^s} = \mathbb{W}_{s+1,t}.$$

A.5. Examples. The elements σ of $\mathbb{S}_{(3,2)}^5$ are

$$(1,2,3,4,5), (1,2,4,3,5), (1,2,4,5,3), (1,4,2,3,5), (1,4,2,5,3), (4,1,2,3,5), \\ (1,4,5,2,3), (4,1,2,5,3), (4,1,5,2,3), (4,5,1,2,3),$$

and the corresponding $\sigma_{\mathcal{C}}$ are given by

$$\text{id}, \Psi_3, \Psi_4 \Psi_3, \Psi_2 \Psi_3, \Psi_2 \Psi_4 \Psi_3, \Psi_1 \Psi_2 \Psi_3, \\ \Psi_3 \Psi_2 \Psi_4 \Psi_3, \Psi_1 \Psi_2 \Psi_4 \Psi_3, \Psi_1 \Psi_3 \Psi_2 \Psi_4 \Psi_3, \Psi_2 \Psi_1 \Psi_3 \Psi_2 \Psi_4 \Psi_3.$$

The sum of these gives $\mathbb{W}_{3,2}$, whose graphical representation is

$$\mathbb{W}_{3,2} = \left| \begin{array}{c} | \\ | \\ | \\ | \\ | \end{array} \right| + \left| \begin{array}{c} | \quad \diagdown \quad | \\ | \quad \diagup \quad | \\ | \quad \diagdown \quad | \\ | \quad \diagup \quad | \end{array} \right| + \left| \begin{array}{c} | \quad \diagdown \quad | \quad \diagdown \quad | \\ | \quad \diagup \quad | \quad \diagup \quad | \\ | \quad \diagdown \quad | \quad \diagdown \quad | \\ | \quad \diagup \quad | \quad \diagup \quad | \end{array} \right| + \left| \begin{array}{c} | \quad \diagdown \quad | \quad \diagdown \quad | \quad \diagdown \quad | \\ | \quad \diagup \quad | \quad \diagup \quad | \quad \diagup \quad | \\ | \quad \diagdown \quad | \quad \diagdown \quad | \quad \diagdown \quad | \\ | \quad \diagup \quad | \quad \diagup \quad | \quad \diagup \quad | \end{array} \right| + \left| \begin{array}{c} | \quad \diagdown \quad | \quad \diagdown \quad | \quad \diagdown \quad | \quad \diagdown \quad | \\ | \quad \diagup \quad | \quad \diagup \quad | \quad \diagup \quad | \quad \diagup \quad | \\ | \quad \diagdown \quad | \quad \diagdown \quad | \quad \diagdown \quad | \quad \diagdown \quad | \\ | \quad \diagup \quad | \quad \diagup \quad | \quad \diagup \quad | \quad \diagup \quad | \end{array} \right|$$

$$+ \left[\begin{array}{c} \diagup \diagdown \\ | \quad | \\ \diagdown \diagup \end{array} \right] + \left[\begin{array}{c} \diagup \diagdown \\ | \quad \diagdown \diagup \\ | \quad | \end{array} \right] + \left[\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ | \quad | \end{array} \right] + \left[\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \diagdown \diagup \end{array} \right]$$

The “braided integers” $\{r+1\} = \mathbb{W}_{1,r}$ are explicitly given by

$$\begin{aligned} \mathbb{W}_{1,1} &= \text{id} + \Psi_1 &= \left[\begin{array}{c} | \\ | \end{array} \right] + \left[\begin{array}{c} \diagup \\ \diagdown \end{array} \right], \\ \mathbb{W}_{1,2} &= \text{id} + \Psi_1 + \Psi_2 \Psi_1 &= \left[\begin{array}{c} | \\ | \\ | \end{array} \right] + \left[\begin{array}{c} \diagup \\ | \\ \diagdown \end{array} \right] + \left[\begin{array}{c} \diagup \diagdown \\ | \end{array} \right], \\ \mathbb{W}_{1,3} &= \text{id} + \Psi_1 + \Psi_2 \Psi_1 + \Psi_3 \Psi_2 \Psi_1 &= \left[\begin{array}{c} | \\ | \\ | \\ | \end{array} \right] + \left[\begin{array}{c} \diagup \\ | \\ | \\ \diagdown \end{array} \right] + \left[\begin{array}{c} \diagup \diagdown \\ | \\ | \end{array} \right] + \left[\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ | \end{array} \right], \end{aligned}$$

and so on, with an obvious pattern. The “mirror integers” are

$$\begin{aligned} \mathbb{W}_{2,1} &= \text{id} + \Psi_2 + \Psi_1 \Psi_2 &= \left[\begin{array}{c} | \\ | \\ | \end{array} \right] + \left[\begin{array}{c} \diagup \\ | \\ \diagdown \end{array} \right] + \left[\begin{array}{c} \diagup \diagdown \\ | \end{array} \right], \\ \mathbb{W}_{3,1} &= \text{id} + \Psi_3 + \Psi_2 \Psi_3 + \Psi_1 \Psi_2 \Psi_3 &= \left[\begin{array}{c} | \\ | \\ | \\ | \end{array} \right] + \left[\begin{array}{c} \diagup \\ | \\ | \\ \diagdown \end{array} \right] + \left[\begin{array}{c} \diagup \diagdown \\ | \\ | \end{array} \right] + \left[\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ | \end{array} \right], \end{aligned}$$

and so on.

A.6. Shuffle identities from braided Hopf algebra structures. We show how the axioms of braided Hopf algebras and their different modules in Sec. 3 and 4 are reformulated as identities in the braid group algebra, in terms of the operators introduced in **A.2–A.4**.

A.6.1. Braided Hopf algebra axioms in terms of shuffles. For the product and co-product in $\mathcal{H}(X)$ introduced in **3.1**, the fundamental Hopf-algebra axiom in Eq. (B.1), read from right to left, is equivalent to the identities

$$\sum_{i=0}^r \sum_{j=0}^s \mathbb{W}_{i,j} \mathbb{W}_{r-i,s-j}^{\uparrow(i+j)} \Psi_{r-i,j}^{\uparrow i} = (r+s+1) \mathbb{W}_{r,s}$$

The two sums in the left-hand side are the two coactions in the right-hand side of (B.1), here in the form of the deconcatenations of (r) into $(i) \otimes (r-i)$ and of (s) into $(j) \otimes (s-i)$; the resulting groups of strands are then braided and shuffle-multiplied as prescribed by the diagram. The numerical factor $r+s+1$ in the right-hand side is what remains in the shuffle language of the coproduct $\Delta : (r+s) \mapsto \sum_{i=0}^{r+s} (i) \otimes (r+s-i)$ in the left-hand side of (B.1)

The defining properties of the antipode in (B.2) are equivalent to the identities

$$\sum_{s=0}^r \mathbb{W}_{s,r-s} \mathcal{S}_s = 0 \quad \text{and} \quad \sum_{s=0}^r \mathbb{W}_{s,r-s} \mathcal{S}_{r-s}^{\uparrow s} = 0 \quad \text{for all } r \geq 1.$$

A.6.2. Hopf bimodule axioms in terms of shuffles. The axioms of Hopf $\mathcal{H}(X)$ -bimodules can also be stated as identities in the braid group algebra. For example, to reformulate the axiom relating the left action and left coaction (the first diagram in **B.3**), we first decompose the left action in (3.3) into a sum

$$(A.6) \quad (r) \cdot (s; Y; t) = \bigoplus_{i=0}^r \mathcal{L}_{i,r,s,t}((r) \otimes (s; Y; t)),$$

where each $\mathcal{L}_{i,r,s,t}$ maps in a particular graded component:

$$\mathcal{L}_{i,r,s,t} = \mathbb{W}_{i,s} \mathbb{W}_{r-i,t}^{\uparrow(i+s+1)} \Psi_{r-i,s+1}^{\uparrow i} : ((r) \otimes (s; Y; t)) \rightarrow (i+s; Y; r-i+t).$$

Then the left–left Hopf module axiom reformulates as

$$\sum_{i=0}^r (i+s+1) \mathcal{L}_{i,r,s,t} = \sum_{i=0}^r \sum_{j=0}^s \mathbb{W}_{i,j} \mathbb{W}_{r-i,s-j+1+t}^{\uparrow(i+j)} \Psi_{r-i,j}^{\uparrow i}.$$

The left-module–right-comodule Hopf axiom is similarly expressed as the identity

$$\sum_{i=0}^r (t+r-i+1) \mathcal{L}_{i,r,s,t} = \sum_{i=0}^r \sum_{j=0}^t \mathbb{W}_{i,s+1+j} \mathbb{W}_{r-i,t-j}^{\uparrow(i+s+1+j)} \Psi_{r-i,s+1+j}^{\uparrow i}.$$

The right action in (3.4) can similarly be represented in the form that explicitly shows the graded components in the target,

$$(A.7) \quad (s; Y; t) \cdot (r) = \bigoplus_{i=0}^r \mathcal{R}_{i,s,t,r}((s; Y; t) \otimes (r)),$$

where

$$\mathcal{R}_{i,s,t,r} = \mathbb{W}_{s,i} \mathbb{W}_{t,r-i}^{\uparrow(s+i+1)} \Psi_{t+1,i}^{\uparrow s} : (s; Y; t) \otimes (r) \rightarrow (s+i; Y; t+r-i)$$

satisfies the identities similar to those for $\mathcal{L}_{i,r,s,t}$.

A.6.3. The Yetter–Drinfeld axiom in terms of shuffles. Referring to the construction of Yetter–Drinfeld modules in **3.3**, we next write the “quantum shuffle” identity that is a braid-algebra translation of the Yetter–Drinfeld axiom satisfied by the adjoint action and the deconcatenation coaction. Diagram (B.7) expressing the Yetter–Drinfeld axiom translates into the shuffle identity

$$\sum_{i=0}^r \sum_{j=0}^{i+s} \mathbb{W}_{j,r-i} \Psi_{i+s-j+1,r-i}^{\uparrow j} \text{Ad}_{i,s} \Psi_{r-i,s+1}^{\uparrow i} = \sum_{i=0}^r \sum_{j=0}^s \mathbb{W}_{i,j} \text{Ad}_{r-i,s-j}^{\uparrow(i+j)} \Psi_{r-i,j}^{\uparrow i}.$$

In the left-hand side here, the two Ψ are the two braidings in the left-hand side of (B.7); the sum over i corresponds to the coproduct and the sum over j to the coaction in the left-hand side of (B.7); $\text{Ad}_{i,s}$ is, clearly, the \hookrightarrow action, and $\mathbb{W}_{j,r-i}$ is the product in the algebra. In the right-hand side, similarly, the sum over i corresponds to the coproduct, the

sum over j to the coaction, and the three operators from right to left are the braiding, the adjoint action, and the product in the right-hand side of (B.7).

APPENDIX B. BRAIDED HOPF ALGEBRAS

For basics on Hopf algebras, we refer the reader to [72, 73]. The main definitions pertaining to braided Hopf algebras [63] (also see [67, 68, 31, 66]) are recalled in what follows.

B.1. Standard notation and definitions.

Eine besondere Bezeichnungsweise mag unwichtig sein,
aber wichtig ist es immer, dass diese eine
mögliche Bezeichnungsweise ist.

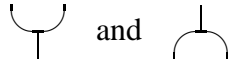
L. WITTGENSTEIN, *Tractatus*, 3.3421

We use the standard graphical notation for braided categories [66]. The braiding $\Psi : \mathcal{X} \otimes \mathcal{Y} \rightarrow \mathcal{Y} \otimes \mathcal{X}$ of two objects \mathcal{X} and \mathcal{Y} in a braided monoidal category \mathcal{C} is denoted as



The diagrams are read from top down. In most cases, \mathcal{X} and \mathcal{Y} are omitted from the notation.

B.1.1. Braided Hopf algebra axioms. For a bialgebra \mathcal{H} in \mathcal{C} , the product $\mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H}$ and coproduct $\mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ are respectively denoted as



The “trademark” of braided Hopf algebras is the braided bialgebra axiom expressing the compatibility condition between the above two maps. It is given by the following identity for maps $\mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$:

(B.1)

The axiom for the antipode $S : \mathcal{H} \rightarrow \mathcal{H}$ is

(B.2)

where \circlearrowleft is the counit and \circlearrowright is the unit of \mathcal{H} . It then follows that

$$\begin{array}{c} \circlearrowleft \\ | \\ \circlearrowright \end{array} = \begin{array}{c} \diagup \diagdown \\ | \quad | \\ \circlearrowleft \quad \circlearrowright \end{array} \quad \text{and} \quad \begin{array}{c} \circlearrowright \\ | \\ \circlearrowleft \end{array} = \begin{array}{c} \circlearrowleft \quad \circlearrowright \\ \diagdown \diagup \\ | \end{array}$$

B.1.2. Modules and comodules. Left module, left comodule, right module, and right comodule structures are expressed as

$$\begin{array}{c} \mathcal{H} \mathcal{X} \\ | \\ \mathcal{X} \end{array} \quad \begin{array}{c} \mathcal{X} \\ | \\ \mathcal{H} \mathcal{X} \end{array} \quad \begin{array}{c} \mathcal{X} \mathcal{H} \\ | \\ \mathcal{X} \end{array} \quad \begin{array}{c} \mathcal{X} \\ | \\ \mathcal{X} \mathcal{H} \end{array}$$

The left module axiom (“associativity” of the action) is

$$\begin{array}{c} \diagup \diagdown \\ | \quad | \\ \diagup \diagdown \end{array} = \begin{array}{c} \diagup \diagdown \\ | \quad | \\ \diagup \diagdown \end{array}$$

The left comodule axiom (“coassociativity” of the coaction) is an upside-down of this, and the right module and right comodule axioms are formulated similarly (mirror symmetrically).

A left–left (left–right) Hopf \mathcal{H} -module in \mathcal{C} is a left module and left (right) comodule with the action and coaction satisfying the respective axiom

$$(B.3) \quad \begin{array}{c} \diagup \diagdown \\ | \quad | \\ \diagup \diagdown \end{array} = \begin{array}{c} \diagup \diagdown \\ | \quad | \\ \diagup \diagdown \end{array} \quad \text{and} \quad \begin{array}{c} \diagup \diagdown \\ | \quad | \\ \diagup \diagdown \end{array} = \begin{array}{c} \diagup \diagdown \\ | \quad | \\ \diagup \diagdown \end{array}$$

Mirror-reflected versions define the Hopf-module axioms for right action and right coaction, and for right action and left coaction:

$$(B.4) \quad \begin{array}{c} \diagup \diagdown \\ | \quad | \\ \diagup \diagdown \end{array} = \begin{array}{c} \diagup \diagdown \\ | \quad | \\ \diagup \diagdown \end{array} \quad \text{and} \quad \begin{array}{c} \diagup \diagdown \\ | \quad | \\ \diagup \diagdown \end{array} = \begin{array}{c} \diagup \diagdown \\ | \quad | \\ \diagup \diagdown \end{array}$$

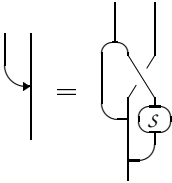
B.1.3. Hopf bimodules. A Hopf bimodule (“bi-Hopf module”) in a braided category \mathcal{C} is a left–left Hopf module that is also a right–right, left–right, and right–left Hopf module, as well as a bimodule and a bicomodule. Hence, both (B.3) and (B.4) hold, and the left action commutes with the right action, and similarly for the coactions.

Hopf bimodules in braided categories were studied in [68, 31], where further references can be found (Hopf bimodules appeared in [26] as “bicovariant bimodules”; also see [64]).

B.1.4. “Relative antipode.” Bespalov’s “relative antipode” [67] is defined for any Hopf bimodule \mathcal{X} as the map $\sigma : \mathcal{X} \rightarrow \mathcal{X}$ given by

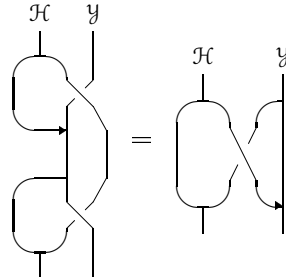
(B.5) 

B.1.5. The “adjoint” action. Any \mathcal{H} -bimodule carries *the (left) “adjoint” action* of \mathcal{H} given by

(B.6)  = $\cdot (\cdot \otimes S)(\text{id} \otimes \Psi)(\Delta \otimes \text{id})$

(where \cdot stands for both left and right actions). The quotation marks in “adjoint” are to remind us that this is not necessarily *the* adjoint action of \mathcal{H} on itself; we omit them in most cases, however.

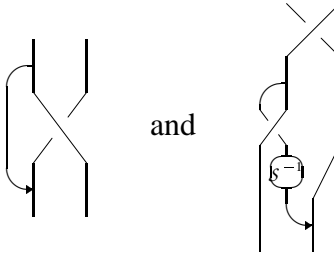
B.1.6. Yetter–Drinfeld modules. In a braided category, a left–left Yetter–Drinfeld (a.k.a. “crossed”) module is a left module under an action $\curvearrowright : \mathcal{H} \otimes \mathcal{Y} \rightarrow \mathcal{Y}$ and left comodule under a coaction $\lrcorner : \mathcal{Y} \rightarrow \mathcal{H} \otimes \mathcal{Y}$ that are related as

(B.7) 

Yetter–Drinfeld modules in braided categories were studied in [67, 68], where further references can be found. They form a braided monoidal category: the action on $\mathcal{Y} \otimes \mathcal{Z}$ is diagonal, which means that

(B.8) 

and the coaction is codiagonal (the upside down of the above diagram). The braiding and the inverse braiding are well known to be respectively given by (see, e.g., [67])

(B.9) 

(we always assume the antipode to be bijective, i.e., S^{-1} to exist).

For a Yetter–Drinfeld module \mathcal{Y} , the “squared relative antipode” [67] is the map $\sigma_2 : \mathcal{Y} \rightarrow \mathcal{Y}$ given by

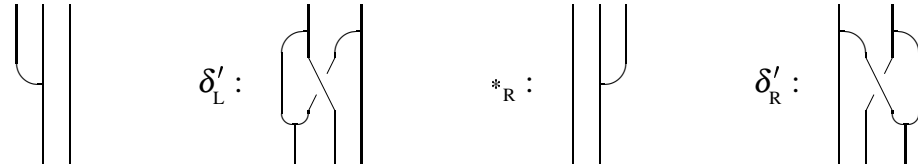
(B.10) 

(see [68] for its properties and use).

B.1.7. Category equivalence. For a Hopf algebra \mathcal{H} in a braided monoidal category with split idempotents, the category ${}^{\mathcal{H}}_{\mathcal{H}}\mathcal{M}$ of Hopf \mathcal{H} -bimodules is braided monoidal equivalent to the category ${}^{\mathcal{H}}_{\mathcal{H}}\mathcal{YD}$ of Yetter–Drinfeld \mathcal{H} -modules. The functor ${}^{\mathcal{H}}_{\mathcal{H}}\mathcal{M} \rightarrow {}^{\mathcal{H}}_{\mathcal{H}}\mathcal{YD}$ is given by taking right coinvariants, and the inverse functor, by induction (see [67, 68] and the references therein, [64] in particular).

B.2. The \times map of Hopf bimodules. As before, we fix a Hopf algebra \mathcal{H} in a braided monoidal category \mathcal{C} and consider Hopf bimodules over \mathcal{H} .

The tensor product of two Hopf bimodules can be given the structure of a Hopf bimodule under the “left” left action, codiagonal left coaction, “right” right action, and codiagonal right coaction, respectively given by

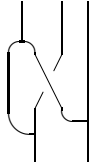
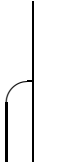
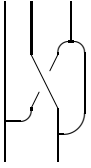

(B.11) 

(see, e.g., [68, 31]).

B.2.1. For any two Hopf bimodules \mathcal{X} and \mathcal{Y} , we define $X(\mathcal{X} \otimes \mathcal{Y})$ as

(B.12) 

The map X intertwines the actions and coactions in (B.11) respectively with the diagonal left action, “left” left coaction, diagonal right action, and “right” right coaction:

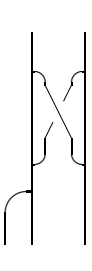
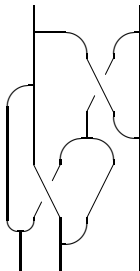
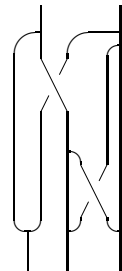
(B.13) $\cdot_L :$  $\delta'_L :$  $\cdot_R :$  $\delta'_R :$ 

i.e.,

$$X \circ *_L = \cdot_L \circ (\text{id} \otimes X), \quad (\text{id} \otimes X) \circ \delta'_L = \delta'_L \circ X,$$

$$X \circ *_R = \cdot_R \circ (X \otimes \text{id}), \quad (X \otimes \text{id}) \circ \delta'_R = \delta'_R \circ X.$$

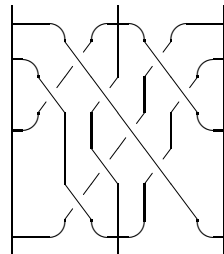
This is shown by simple manipulations with the diagrams. For example, to prove the second identity, we start with its right-hand side and use the relevant Hopf-module axiom and then the coaction property and the commutativity of left and right coactions:

(B.14) $\delta'_L \circ X =$  $=$  $=$  $= (\text{id} \otimes X) \circ \delta'_L.$

B.2.2. It follows that X is associative in the following sense:

$$X(X(\mathcal{X} \otimes \mathcal{Y}) \otimes \mathcal{Z}) = X(\mathcal{X} \otimes X(\mathcal{Y} \otimes \mathcal{Z})),$$

where *each of the inner $X(U, V)$ is understood as a Hopf bimodule under $(\cdot_L, \delta'_L, \cdot_R, \delta'_R)$* . This is also readily verified diagrammatically, with both sides of the last formula being given by

 where $\cap = \cap$ and $\Psi = \Psi$

We can therefore define $X(\mathcal{X} \otimes \mathcal{Y} \otimes \mathcal{Z} \otimes \dots)$ by nesting.

B.3. The \mathcal{U} ($[1:]$) map of right coinvariants. For right coinvariants in Hopf bimodules, X in (B.12) reduces to the map $\mathcal{U} : \mathcal{X} \otimes \mathcal{Y} \rightarrow \mathcal{X} \otimes \mathcal{Y}$ given by the diagram

$$\mathcal{U} : \begin{array}{c} | \\ \text{---} \text{---} \text{---} \\ | \end{array}$$

which yields right coinvariants and which inherits the property to intertwine δ'_L and δ_L , i.e.,

$$(B.15) \quad \begin{array}{c} \text{---} \text{---} \text{---} \\ | \end{array} = \begin{array}{c} | \\ \text{---} \text{---} \text{---} \\ | \end{array}$$

as an immediate application of a Hopf-module axiom, with both sides being equal to

$$\begin{array}{c} \text{---} \text{---} \text{---} \\ | \end{array}.$$

We also have $\mathcal{U} \circ *_R = \cdot_R \circ (\mathcal{U} \otimes \text{id})$. In verifying the associativity, we then find that

$$(B.16) \quad \mathcal{U}(\mathcal{U}(\mathcal{X} \otimes \mathcal{Y}) \otimes \mathcal{Z}) = \mathcal{U}(\mathcal{X} \otimes \mathcal{U}(\mathcal{Y} \otimes \mathcal{Z})) = \begin{array}{c} | \\ \text{---} \text{---} \text{---} \\ | \end{array}$$

This allows defining the action of \mathcal{U} on multiple tensor products of right coinvariants in Hopf bimodules by nesting.

Right coinvariants in Hopf bimodules can be viewed as Yetter–Drinfeld modules under the adjoint action. The \mathcal{U} map intertwines the diagonal action on tensor products, Eq. (B.8), as follows:

$$(B.17) \quad \begin{array}{c} \text{---} \text{---} \text{---} \\ | \end{array} = \begin{array}{c} \text{---} \text{---} \text{---} \\ | \end{array}$$

This is shown by direct calculation in Fig. B.1. Everything below the top \mathcal{U} -diagram in

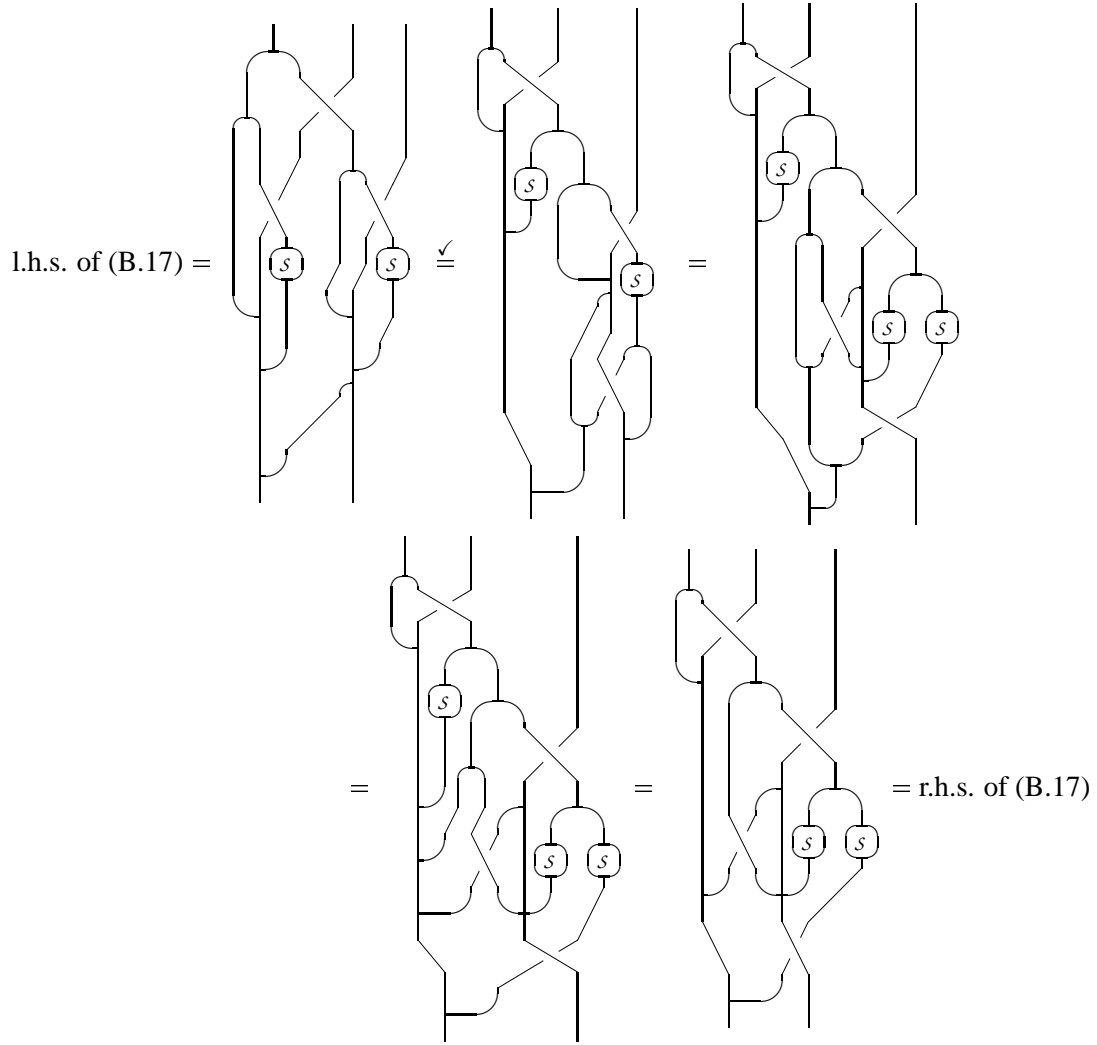
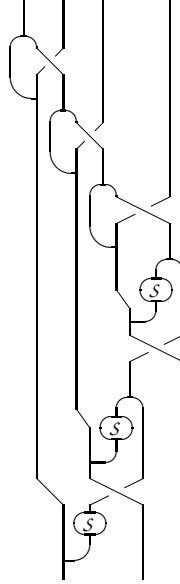


FIGURE B.1. Proof of (B.17). The \cong equality and the next one use two Hopf-module axioms and the braided anti-automorphism property of the antipode. The equality connecting the two lines of the formula involves the right action associativity property, and the next equality uses coassociativity to isolate a “bubble” as in (B.2), which is then eliminated.

the right-hand side of (B.17) shows how the Hopf algebra acts on $\mathcal{U}(\mathcal{X} \otimes \mathcal{Y})$.

The pattern extends to multiple tensor products in the manner that is entirely clear from the example of a triple product: the \mathcal{U} map “squared,” i.e., *the diagram in (B.16)*,

intertwines the diagonal adjoint action on a triple tensor product with the action given by



The extension to n -fold tensor products is now entirely obvious (and gives a fully braided version of the “unexpected” action in [74]). The action on the *last*, rightmost tensor factor is the adjoint in all cases.

APPENDIX C. YETTER–DRINFELD BRAIDING AND DIAGONAL BRAIDING

An important class of braidings is provided by Yetter–Drinfeld categories.

Let H be an ordinary Hopf algebra (with bijective antipode s). A left–left Yetter–Drinfeld H -module R is by definition a left H -module (with an action $H \otimes R \rightarrow R : h \otimes r \mapsto h \triangleright r$) and left H -comodule (with coaction $R \rightarrow H \otimes R : r \mapsto r_{(-1)} \otimes r_{(0)}$) satisfying the defining axiom

$$(h' \triangleright r)_{(-1)} h'' \otimes (h' \triangleright r)_{(0)} = h' r_{(-1)} \otimes h'' \triangleright r_{(0)}$$

(which is just (B.7) with the braiding given by transposition), or equivalently

$$(C.1) \quad (h \triangleright r)_{(-1)} \otimes (h \triangleright r)_{(0)} = h' r_{(-1)} s(h''') \otimes h'' \triangleright r_{(0)}.$$

The category ${}^H_H\mathcal{YD}$ of Yetter–Drinfeld H -modules is monoidal and braided, with the braiding and the inverse given by (cf. (B.9))

$$(C.2) \quad \Psi : \mathcal{U} \otimes \mathcal{V} \rightarrow \mathcal{V} \otimes \mathcal{U} : u \otimes v \mapsto u_{(-1)} \triangleright v \otimes u_{(0)},$$

$$(C.3) \quad \Psi^{-1} : \mathcal{V} \otimes \mathcal{U} \rightarrow \mathcal{U} \otimes \mathcal{V} : v \otimes u \mapsto v_{(0)} \otimes s^{-1}(v_{(-1)}) \triangleright u.$$

A Hopf algebra \mathcal{R} in ${}^H_H\mathcal{YD}$ is a Yetter–Drinfeld H -module and a braided Hopf algebra whose braiding is that of ${}^H_H\mathcal{YD}$ and whose operations are ${}^H_H\mathcal{YD}$ morphisms (this notion can be defined purely categorically).

Any braided Hopf algebra whose braiding is rigid can be realized as a Hopf algebra in ${}^H_H\mathcal{YD}$ for some ordinary Hopf algebra H (which is by far not unique) [39].

For a Hopf algebra \mathcal{R} in ${}^H_H\mathcal{YD}$, we consider left \mathcal{R} modules and comodules $\mathcal{Y} \in {}^H_H\mathcal{YD}$. Because any such \mathcal{Y} is also an H module and comodule, we have to clearly distinguish the different actions and coactions. We write $r \rightarrow y$ and $h.y$ for the \mathcal{R} and H actions, and $y \mapsto y_{\underline{-1}} \otimes y_{\underline{0}} \in \mathcal{R} \otimes \mathcal{Y}$ and $y \mapsto y_{(-1)} \otimes y_{(0)} \in H \otimes \mathcal{Y}$ for the coactions. That the \mathcal{R} -coaction is an H -comodule morphism is then expressed as

$$(C.4) \quad y_{(-1)} \otimes y_{(0)\underline{-1}} \otimes y_{(0)\underline{0}} = y_{\underline{-1}(-1)} y_{\underline{0}(-1)} \otimes y_{\underline{-1}(0)} \otimes y_{\underline{0}(0)} \in H \otimes \mathcal{R} \otimes \mathcal{Y}.$$

It is also an H -module morphism: $(h.y)_{\underline{-1}} \otimes (h.y)_{\underline{0}} = (h' \triangleright y_{\underline{-1}}) \otimes h''.y$.

If we assume that the braided linear spaces X and Y in Secs. 3 and 4 are objects in ${}^H_H\mathcal{YD}$, then Ψ is given by (C.2) and there is an H -coaction $x \mapsto x_{(-1)} \otimes x_{(0)}$ on X , and similarly on Y , and then

$$(x_1 \otimes \dots \otimes x_n)_{(-1)} \otimes (x_1 \otimes \dots \otimes x_n)_{(0)} = x_{1(-1)} \dots x_{n(-1)} \otimes (x_{1(0)} \otimes \dots \otimes x_{n(0)}).$$

Also, $(\)^{\underline{-1}} \otimes (\)^{\underline{2}}$ and $(\)_{\underline{-1}} \otimes (\)_{\underline{0}}$ are now the deconcatenation coproduct and coaction:

$$(x_1 \otimes \dots \otimes x_n)^{\underline{-1}} \otimes (x_1 \otimes \dots \otimes x_n)^{\underline{2}} = \sum_{i=0}^n (x_1 \otimes \dots \otimes x_i) \otimes (x_{i+1} \otimes \dots \otimes x_n)$$

and

$$(x_1 \otimes \dots \otimes x_n \otimes y)_{\underline{-1}} \otimes (x_1 \otimes \dots \otimes x_n \otimes y)_{\underline{0}} = \sum_{i=0}^n (x_1 \otimes \dots \otimes x_i) \otimes (x_{i+1} \otimes \dots \otimes x_n \otimes y).$$

The adjoint action

$$\text{Ad}_{1,s} : X \otimes X^{\otimes s} \otimes Y \rightarrow X^{\otimes(s+1)} \otimes Y$$

then evaluates as (for $x \in X$ and $b \in X^{\otimes s} \otimes Y$)

$$\text{Ad}_{1,s}(x \otimes b) = x_{(-1)} \triangleright b_{\underline{-1}} \otimes x_{(0)} \otimes b_{\underline{0}} - x_{(-1)}' \triangleright (b_{\underline{-1}} \otimes b_{\underline{0}(-1)} s(x_{(-1)}'') \triangleright x_{(0)} \otimes b_{\underline{0}(0)}).$$

Diagonal braiding in Yetter–Drinfeld categories is associated with commutative co-commutative H , i.e., group algebras of (finite) Abelian groups. This case has been considered in numerous papers, and in particular in the “Nichols” context—in the framework of Andruskiewitsch and Schneider’s classification program ([13, 14], and the references therein). We follow these papers in briefly recalling the relevant points about diagonal braiding.

For a finite group Γ , left Yetter–Drinfeld $k\Gamma$ -modules are Γ -graded vector spaces $X = \bigoplus_{g \in \Gamma} X_g$ with a Γ action such that $gX_h \subset X_{ghg^{-1}}$; the left comodule structure is here given by $\delta x = g \otimes x$ for all $x \in X_g$. If, moreover, Γ is Abelian, then Yetter–Drinfeld $k\Gamma$ -modules are just Γ -graded vector spaces with an action of Γ on each X_g . The action is then

diagonalizable, and hence $X = \bigoplus_{\chi \in \hat{\Gamma}} X^\chi$, where $X^\chi = \{x \in X \mid gx = \chi(g)x \text{ for all } g \in \Gamma\}$ and $\hat{\Gamma}$ is the group of characters of Γ . Then

$$X = \bigoplus_{g \in \Gamma, \chi \in \hat{\Gamma}} X_g^\chi,$$

where $X_g^\chi = X^\chi \cap X_g$. Hence, each Yetter–Drinfeld $k\Gamma$ -module X has a basis (x_i) such that, for some $g_i \in \Gamma$ and $\chi_i \in \hat{\Gamma}$, $\delta x_i = g_i \otimes x_i$ and $gx_i = \chi_i(g)x_i$ for all g . The braiding (C.2) then takes the form (5.1) with $q_{ij} = \chi_j(g_i)$.⁹

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⁹Diagonal braiding is also said to be of the Abelian group type in the terminology in [13]. Whenever the q_{ij} are roots of unity, the braiding is of the *finite group type*, which means that the group generated by the g_i in $\text{End} X$ is finite.

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